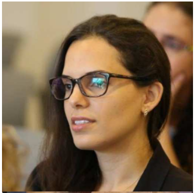
Open Bar: a Brouwerian Intuitionistic Logic with a Pinch of Excluded Middle

Mark Bickford, Liron Cohen, Bob Constable, Vincent Rahli

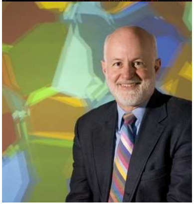
January 27, 2021

In collaboration with



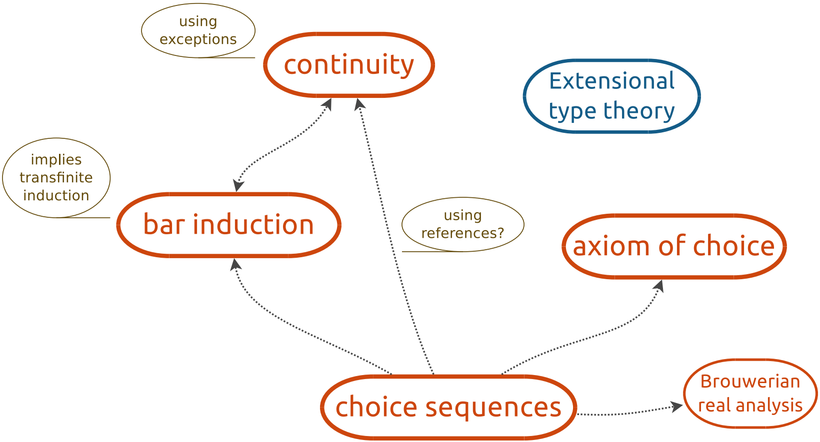
Liron Cohen

Mark Bickford



Bob Constable

Towards a Brouwerian Intuitionistic Logic



Towards a Brouwerian Intuitionistic Logic

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- Contradict classical axioms!
- Enough to validate Bar Induction?

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- Contradict classical axioms!
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2020: (OpenTT) Relaxed model to validate classical axioms

- Consistent with classical axioms!
- Enough to validate Bar Induction?

Starting point: an Extensional Type Theory

Untyped call-by-name lambda-calculus

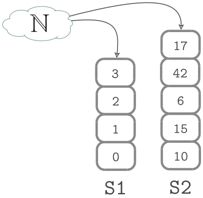
Introduction/elimination deduction rules

realizability semantics

- Extensional
- Dependent types
- Partial types

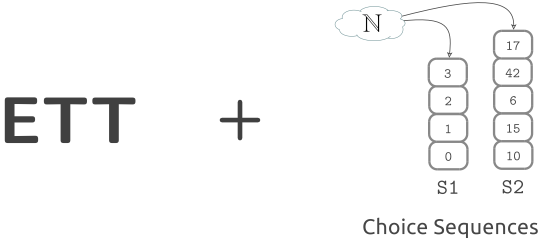
Adding Choice Sequences

ETT +



Choice Sequences

Adding Choice Sequences



Broader sense of computation

Adding Choice Sequences

lawless (free choice) sequences: no restrictions on the choices (except for initial segments)

LS$_1$ (density) $\forall s.\exists\alpha.\alpha \in s$

LS$_2$ (discreteness) $\forall\alpha,\beta.(\alpha \equiv \beta \ \vee \ \neg\alpha \equiv \beta)$

LS$_3$ (open data) $A(\alpha) \ \Rightarrow \ \exists n.\forall\beta.(\overline{\alpha}n = \overline{\beta}n \ \Rightarrow \ A(\beta))$

s	finite sequence
α	lawless sequence
$\alpha \in s$	s is an initial segment of α
$\equiv$	intensional equality
$\overline{\alpha}n$	the initial segment of α of length n

Adding Choice Sequences

BITT

LS1 $\Pi n{:}\mathbb{N}.\Pi f{:}\mathcal{B}_n.\Sigma\alpha{:}\texttt{Free}.f = \alpha \in \mathcal{B}_n$

LS2 $\Pi\alpha, \beta{:}\texttt{Free}.(\alpha{=}\beta \in \mathcal{B}){+}(\neg\alpha{=}\beta \in \mathcal{B})$

LS3 –

¬LEM $\neg\Pi P{:}\mathbb{P}.{\downarrow}(P{+}\neg P)$

¬MP $\neg\Pi P{:}\mathbb{B}^{\mathbb{N}}.\neg(\Pi n{:}\mathbb{N}.\neg P(n)) \rightarrow \Sigma n{:}\mathbb{N}.P(n)$

¬IP $\neg\Pi A{:}\mathbb{P}.\Pi B{:}\mathbb{P}^{\mathbb{N}}.(A \rightarrow \Sigma n{:}\mathbb{N}.B(n)) \rightarrow \Sigma n{:}\mathbb{N}.(A \rightarrow B(n))$

¬LPO $\neg\Pi P{:}\mathbb{B}^{\mathbb{N}}.(\Sigma\mathbb{N}{:}n.P(n)){+}(\Pi n{:}\mathbb{N}.\neg P(n))$

OpenTT

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LS3 $\Pi\alpha{:}\texttt{Free}.P(\alpha) \rightarrow \Sigma n{:}\mathbb{N}_{\wr}.\Pi\beta{:}\texttt{Free}.(\alpha = \beta \in \mathcal{B}_{\wr n} \rightarrow {\downarrow}P(\beta))$

LEM $\Pi P{:}\mathbb{P}.{\downarrow}(P{+}\neg P)$

(where $\mathcal{B} = \mathbb{N}^{\mathbb{N}}$ and $\mathcal{B}_n = \mathbb{N}^{\mathbb{N}_n}$)
($\downarrow$ is a "proof erasure" operator)

Syntax & Operational Semantics

Syntax:

$$
\begin{aligned}
T \in \texttt{Type} ::= {} & \mathbb{N} \mid \mathbb{U}_i \mid \mathbf{\Pi} x{:}t.t \mid \mathbf{\Sigma} x{:}t.t \mid \{x : t \mid t\} \\
\mid {} & t = t \in t \mid t{+}t \mid \ldots \\
\mid {} & \texttt{Free} \text{ (choice sequence type)}
\end{aligned}
$$

$$
\begin{aligned}
v \in \texttt{Value} ::= {} & T \mid \star \mid \underline{n} \mid \lambda x.t \mid \langle t, t \rangle \mid \texttt{inl}(t) \mid \texttt{inr}(t) \mid \ldots \\
\mid {} & \eta \text{ (choice sequence name)}
\end{aligned}
$$

$$
\begin{aligned}
t \in \texttt{Term} ::= {} & x \mid v \mid t\ t \mid \texttt{fix}(t) \mid \texttt{let } x := t \texttt{ in } t \\
\mid {} & \texttt{case } t \texttt{ of inl}(x) \Rightarrow t \texttt{ | inr}(y) \Rightarrow t \\
\mid {} & \texttt{let } x, y = t \texttt{ in } t \mid \texttt{if } t{=}t \texttt{ then } t \texttt{ else } t \mid \ldots
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\end{array}
$$

Operational semantics:

$$
\begin{array}{lcl}
(\lambda x.t_1)\ t_2 & \longmapsto & t_1[x \backslash t_2] \\
\texttt{let}\ x_1, x_2 = \langle t_1, t_2 \rangle\ \texttt{in}\ t & \longmapsto & t[x_1 \backslash t_1; x_2 \backslash t_2] \\
\texttt{fix}(v) & \longmapsto & v\ \texttt{fix}(v)
\end{array}
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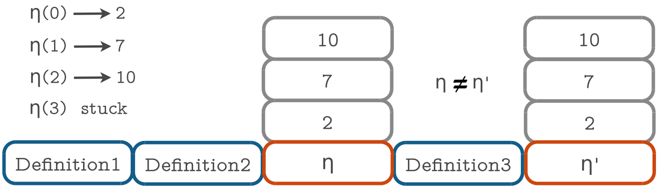
...

World-Based Computations

World-dependent operational semantics:



Worlds include:

- definitions
- choice sequences

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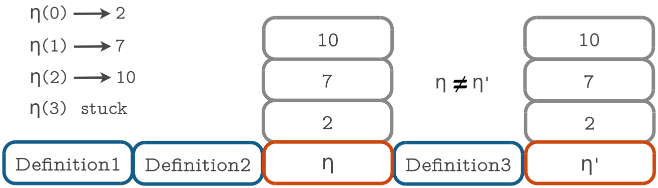

Worlds include:

- definitions
- choice sequences

Worlds can be extended

- horizontally
- vertically
- $w_2 \succeq w_1$

OpenTT

Standard ETT rules:

$$\frac{\Gamma, x : A \vdash b : B[x] \qquad \Gamma \vdash \star : (A \in \mathbb{U}_i)}{\Gamma \vdash \lambda x.b : \mathbf{\Pi} a{:}A.B[a]} \qquad \cdots$$

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- How do we validate these rules?

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+ **choice sequence rules**:

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+ **LS1 (density), LS2 (discreteness), LS3 (Open Data)**

+ **LEM**

- How do we validate these rules?
- Why is LEM an OpenTT rule, but not a BITT rule?

Realizability semantics

An inductive relation that expresses type equality

$T_1 \equiv T_2$ $\qquad\qquad$ $\texttt{type}(T)$ is $T \equiv T$

A recursive function that expresses equality in a type

$$a \equiv b \in T$$

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For example (product types):

$f_1 \equiv f_2 \in \mathbf{\Pi} x{:}A.B$ $\qquad$ $\texttt{type}((\mathbf{\Pi} x{:}A.B)) \land$
$\forall a_1, a_2.\ a_1 \equiv a_2 \in A \Rightarrow f_1(a_1) \equiv f_2(a_2) \in B[x \backslash a_1]$

$\mathbf{\Pi} x_1{:}A_1.B_1 \equiv \mathbf{\Pi} x_2{:}A_2.B_2$ $\qquad$ $A_1 \equiv A_2 \land$
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Enough to validate choice sequence rules?

Going back to BITT for a minute

Why is $\eta \in \mathcal{B}$ valid in BITT?

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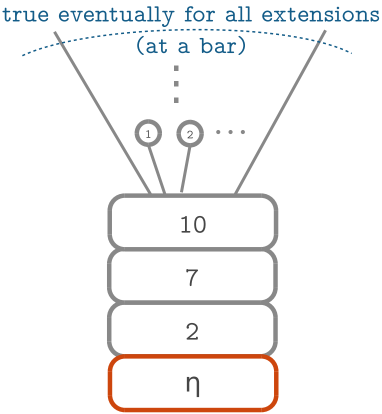
We used a Beth interpretation:



Formally:

$\eta \in \mathcal{B}$ **is true in world** w

$\iff$

$\forall m : \textbf{nat}.\exists b : \text{bar}(w).\forall w' \in b.$
$\eta(m)$ **computes to a nat in** w'

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This model rules out a number of axioms (e.g., LEM)

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Why?

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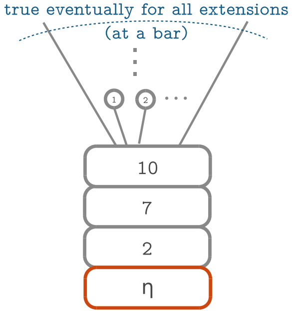
Why?

Given a "fresh" (no choices so far) free choice sequence α,

- it is not true that $\boldsymbol{\Sigma} n{:}\mathbb{N}.\alpha(n) = 1 \in \mathbb{N}$ because there is a path where α is only extended with 0
- it is not true that $\neg\boldsymbol{\Sigma} n{:}\mathbb{N}.\alpha(n) = 1 \in \mathbb{N}$ because there is path where α is extended with 1
  - The meaning of $\neg T$ is that T is false in all extensions

Any way around this?

Beth model

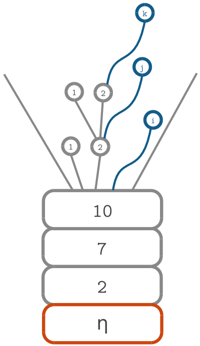


$T@w$

$\iff$

$\exists b \in \mathsf{bar}(w).\forall w_1 \in b.$
$\forall w_2 \succeq w_1.T@w_2$

Open Bar model



$T@w$

$\iff$

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Bears a resemblance to double negation translation:

- Kripke interpretation of $A \rightarrow B$:

$$[\![A \rightarrow B]\!]_w = \forall w_1 \succeq w.[\![A]\!]_{w_1} \Rightarrow [\![B]\!]_{w_1}$$

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$$\forall w_1 \succeq w.\neg\forall w_2 \succeq w_1.\neg[\![A]\!]_{w_2}$$

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- classically equivalent to:

$$\forall w_1 \succeq w.\exists w_2 \succeq w_1.[\![A]\!]_{w_2}$$

LEM in the Open Bar model

This model still satisfies the choice sequence axioms

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$$\frac{}{\Gamma \vdash \lambda P.\star : \mathbf{\Pi} P{:}\mathbb{P}.{\downarrow}(P + \neg P)}$$

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- Let w_0 be the current world
- $\forall w_1 \succeq w_0$, using classical reasoning we can assume that
  - either $\exists w_2 \succeq w_1.P@w_2$
  - or $\neg\exists w_2 \succeq w_1.P@w_2$

  Either way we conclude trivially

Choice sequences in the Open Bar model

LS1 (density) is valid

$$\frac{}{\Gamma \vdash \lambda n, f.\star : \mathbf{\Pi} n{:}\mathbb{N}.\mathbf{\Pi} f{:}\mathcal{B}_n.\downarrow\mathbf{\Sigma}\alpha{:}\texttt{Free}.f = \alpha \in \mathcal{B}_n}$$

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LS2 (discreteness) is valid

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LS3 (Open Data) is valid

$$\begin{array}{l} \hline \Gamma \vdash \lambda\alpha, p.\star : \Pi\alpha{:}\texttt{Free}. \\ \qquad\qquad P(\alpha) \\ \qquad\qquad \rightarrow \downarrow\Sigma n{:}\mathbb{N}.\Pi\beta{:}\texttt{Free}.(\alpha = \beta \in \mathcal{B}_n \rightarrow \downarrow P(\beta)) \end{array}$$

Open Data in the Open Bar model

$$\boldsymbol{\Pi}\alpha{:}\texttt{Free}.P(\alpha) \rightarrow {\downarrow}\boldsymbol{\Sigma}n{:}\mathbb{N}.\boldsymbol{\Pi}\beta{:}\texttt{Free}.(\alpha = \beta \in \mathcal{B}_n \rightarrow {\downarrow}P(\beta))$$

Why is LS3 (Open Data) valid?

Open Data in the Open Bar model

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Why is LS3 (Open Data) valid?

- no computational content: $\lambda\alpha, p.\star$
- we assume $P(\alpha)\texttt{@}w$, where w is the current world
- **within the metatheory we realize the modulus of continuity n with $|w|$ (w's depth)**

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- we assume $P(\alpha)@w$, where w is the current world
- **within the metatheory we realize the modulus of continuity n with $|w|$ (w's depth)**
- we get to assume that α and β have the same choices up to $|w|$ in some $w_1 \succeq w$, and we have to show $P(\beta)@w_1$
- there must be a world $w_1 \succeq w_0 \succeq w$ such that α and β have exactly the same choices in w_0
- by monotonicity: $P(\alpha)@w_0$
- we swap α and β: $P(\beta)@w_0$
- by monotonicity: $P(\beta)@w_1$

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Can we validate a version with computational content?

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Can we compute n solely based on α and $P(\alpha)$?

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- We add an operator to the language to compute the depth of the current world:

$$t \in \texttt{Term} ::= \cdots \mid \texttt{wDepth}$$

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Can we compute n solely based on α and $P(\alpha)$?

At least:

$$\Pi\alpha{:}\texttt{Free}.P(\alpha) \rightarrow \Sigma n{:}\mathbb{N}_{\rightsquigarrow}.\Pi\beta{:}\texttt{Free}.(\alpha = \beta \in \mathcal{B}_{\rightsquigarrow n} \rightarrow \downarrow P(\beta))$$

- We add an operator to the language to compute the depth of the current world:

$$t \in \texttt{Term} ::= \cdots \mid \texttt{wDepth}$$

- We realize this formula using $\lambda\alpha, p.\langle\texttt{wDepth}, \ldots\rangle$

Open Data in the Open Bar model

2 variants of Open Data:

- $\mathbf{\Pi}\alpha{:}\mathtt{Free}.P(\alpha) \rightarrow \downarrow\mathbf{\Sigma}n{:}\mathbb{N}.\mathbf{\Pi}\beta{:}\mathtt{Free}.(\alpha = \beta \in \mathcal{B}_n \rightarrow \downarrow P(\beta))$
- $\mathbf{\Pi}\alpha{:}\mathtt{Free}.P(\alpha) \rightarrow \mathbf{\Sigma}n{:}\mathbb{N}_{\wr\downarrow}.\mathbf{\Pi}\beta{:}\mathtt{Free}.(\alpha = \beta \in \mathcal{B}_{\wr\downarrow n} \rightarrow \downarrow P(\beta))$

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Time squashing vs. space squashing:

- a member of $\mathbb{N}$ computes to the same value in all worlds
- a member of $\mathbb{N}_{\rightsquigarrow}$ can compute to $\neq$ values in $\neq$ worlds

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Also useful to assign types to references

Next. . .

- Can we still validate Bar Induction using such choice sequences?

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- Can we validate continuity using references (similar to such choice sequences)?

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