Realizing Continuity Using Stateful Computations

Liron Cohen and Vincent Rahli

February, 2023

Motivation

Continuity is a key component of intuitionistic logic

$$\forall F : \mathscr{B} \rightarrow \mathbb{N}.\ \forall \alpha : \mathscr{B}.\ \exists n : \mathbb{N}.\ \forall \beta : \mathscr{B}.$$
$$(\alpha = \beta \in \mathscr{B}_n) \rightarrow (F(\alpha) = F(\beta) \in \mathbb{N})$$

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Used for example to prove that all real-valued functions on the unit interval are continuous.

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⚠ **Non-extensional** (Kreisel, Troesltra, Escardó and Xu)

For example: do $\lambda\alpha.0$ and $\lambda\alpha.\texttt{let}\ x = \alpha(10)\ \texttt{in}\ x - x$ have the same modulus of continuity?

This talk in 1 slide

type modality

- Beth
- Kripke
- open

CTT $\longrightarrow$ $TT^{\square}_{C}$ ← choice operator

- reference cells
- choice sequences

Realized continuity using exceptions

Realized continuity using stateful computations

- non-empty modality
- choices are numbers
- choices can be retrieved

(Syntactical account of the semantical methods)

$TT_{\mathscr{C}}^{\square}$: A Family of Extensional Type Theories

A family of extensional type theories parameterized by
a type modality $\square$, and a choice type $\mathscr{C}$,
compatible with intuitionistic and classical principles

Formalized in Agda

$TT^{\square}_{\mathscr{C}}$: A Family of Extensional Type Theories

Untyped call-by-name lambda-calculus

sequent calculus

realizability semantics

Extensional

Dependent types

$TT_{\mathscr{C}}^{\square}$: Syntax

Core Syntax:

$$T \in \texttt{Type} ::= \mathbb{N} \mid \mathbb{U}_i \mid \mathbf{\Pi} x{:}t.t \mid \mathbf{\Sigma} x{:}t.t \mid t = t \in t \mid t{+}t \mid \ldots$$

$$v \in \texttt{Value} ::= T \mid \star \mid \underline{n} \mid \lambda x.t \mid \langle t, t \rangle \mid \texttt{inl}(t) \mid \texttt{inr}(t) \mid \ldots$$

$$\begin{aligned} t \in \texttt{Term} ::= {} & x \mid v \mid t\ t \mid \texttt{fix}(t) \mid \texttt{let}\ x := t\ \texttt{in}\ t \\ & \mid \texttt{case}\ t\ \texttt{of}\ \texttt{inl}(x) \Rightarrow t\ \mathbf{|}\ \texttt{inr}(y) \Rightarrow t \\ & \mid \texttt{let}\ x, y = t\ \texttt{in}\ t \mid \texttt{if}\ t{=}t\ \texttt{then}\ t\ \texttt{else}\ t \mid \ldots \end{aligned}$$

$\mathrm{TT}^{\square}_{\mathscr{C}}$: World-Based Computations

Core Operational Semantics:

$$
\begin{array}{lllcl}
w & \vdash & (\lambda x.t_1)\ t_2 & \longmapsto & t_1[x\backslash t_2] \\
w & \vdash & \mathtt{let}\ x_1,x_2 = \langle t_1,t_2\rangle\ \mathtt{in}\ t & \longmapsto & t[x_1\backslash t_1; x_2\backslash t_2] \\
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where $w \in \mathscr{W}$ (a poset with ordering $\sqsubseteq$)

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So far we haven't used the world

$TT_{\mathscr{C}}^{\square}$: Choice Operator

Additional Components

- $\mathscr{N}$: abstract type of choice names
- $\mathscr{C}$: abstract type of choices inhabited by $\kappa_0 \neq \kappa_1$
- a partial function: $\text{choice} \in \mathscr{W} \rightarrow \mathscr{N} \rightarrow \mathscr{C}$

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Syntax

$$v \in \texttt{Value} ::= \cdots \mid \delta \text{ (choice name)}$$

$$t \in \texttt{Term} ::= \cdots \mid !t \text{ (reading)}$$

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Operational Semantics

$$w \vdash !\delta \mapsto \mathsf{choice}(w,\delta)$$

$TT_{\mathscr{C}}^{\square}$: Inference Rules

Standard ETT rules:

$$\frac{\Gamma, x : A \vdash b : B[x] \qquad \Gamma \vdash \star : (A \in \mathbb{U}_i)}{\Gamma \vdash \lambda x.b : \mathbf{\Pi} a{:}A.B[a]} \qquad \cdots$$

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$TT^{\square}_{\mathscr{C}}$: Realizability semantics

An inductive relation that expresses type equality

$$w \vDash T_1 \equiv T_2$$

A recursive function that expresses equality in a type

$$w \vDash a \equiv b \in T$$

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For example (product types):

$$w \vDash \mathbf{\Pi} x_1{:}A_1.B_1 \equiv \mathbf{\Pi} x_2{:}A_2.B_2$$
$$\iff$$
$$\forall w' \sqsupseteq w. w' \vDash A_1 \equiv A_2 \ \wedge$$
$$\forall w' \sqsupseteq w. \forall a_1, a_2.\ w' \vDash a_1 \equiv a_2 \in A_1 \Rightarrow w' \vDash B_1[x_1 \backslash a_1] \equiv B_2[x_2 \backslash a_2]$$

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An abstract modality on (the semantics of) types: $\square$

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Forcing interpretation: $\square_w(w'.w' \vDash T) \rightarrow w \vDash T$

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Properties (where $(w : \mathscr{W}), (P, Q : \mathscr{P}_w)$**):**

monotonicity of $\Box$	$\forall w' \sqsupseteq w.\Box_w P \rightarrow \Box_{w'} P$
K, distribution axiom	$\Box_w(P \rightarrow Q) \rightarrow \Box_w P \rightarrow \Box_w Q$
$C4$, i.e., $\Box\Box \rightarrow \Box$	$\Box_w(w'.\Box_{w'} P) \rightarrow \Box_w P$
$\forall \rightarrow \Box$	$\forall^{\sqsubseteq}_w(P) \rightarrow \Box_w P$
T, reflexivity axiom	$\forall(P : \mathbb{P}).\Box_w(w'.P) \rightarrow P$

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$\forall \to \square$	$\forall_w^{\sqsubseteq}(P) \to \square_w P$
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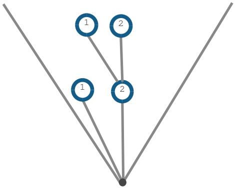
Enough to prove standard properties of the type system: consistency, symmetry, transitivity, etc.

$\mathrm{TT}^{\square}_{\mathscr{C}}$: Examples of Modalities

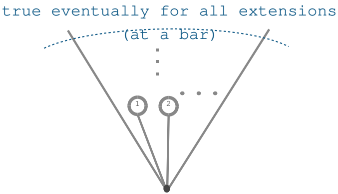
Kripke modality



$$w \vDash T \iff \forall w_1 \sqsupseteq w.w_1 \vDash T$$

(modality: $\square_K(T)$)

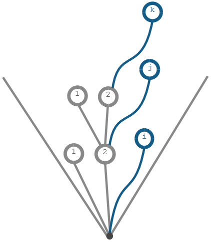
Beth modality



$$w \vDash T \iff \exists b \in \mathrm{bar}(w).\forall w_1 \in b.\ \forall w_2 \sqsupseteq w_1.w_2 \vDash T$$

(modality: $\square_B(T)$)

Open modality



$$w \vDash T \iff \forall w_1 \sqsupseteq w.\exists w_2 \sqsupseteq w_1.\ \forall w_3 \sqsupseteq w_2.w_2 \vDash T$$

(modality: $\square_O(T)$)

$\mathsf{TT}_{\mathscr{C}}^{\Box}$: Modalities from Coverings

Modalities can be derived from **coverings**

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- it is closed under binary intersections, union & subsets
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For example, Kripke, Beth, Open coverings

Continuity – Functions in $\mathbb{N}^{\mathscr{B}}$ only need initial segments

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Continuity axiom for numbers:

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Uniform continuity theorem ($f \in [\alpha, \beta] \rightarrow \mathbb{R}$):

$$\forall \epsilon > 0. \exists \delta > 0. \forall x, y : [\alpha, \beta].\ |x - y| \leq \delta \rightarrow |f(x) - f(y)| \leq \epsilon$$

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Consistent with CTT if truncated (MSCS'17):

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Essence: Test whether a seq. α of length n is long enough

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let exception e in
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 true) handle e ⇒ false
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Plus a loop until the modulus of continuity is reached

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Also consistent with $\mathsf{TT}_{\mathscr{C}}^{\square}$ (CSL'23):

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Again following Longley's method:

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let r = ref 0 in
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```

More straightforward; No need for a loop

Continuity – Purity

Different moduli in extensions:

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- α might get applied to 0 in w_1
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? Are impure functions continuous?

? Can the modulus of continuity inhabit a variant of $\mathbb{N}$ where numbers are allowed to change in extensions?

Continuity – Purity

We require here functions to be pure (Π_p):

Theorem (Continuity Principle)

The following continuity principle, is valid w.r.t. the above semantics:

$$\Pi_p F{:}\mathcal{B} \to \mathbb{N}.\Pi_p \alpha{:}\mathcal{B}.\downharpoonleft \Sigma n{:}\mathbb{N}.\Pi_p \beta{:}\mathcal{B}.$$
$$(\alpha = \beta \in \mathcal{B}_n) \to (F(\alpha) = F(\beta) \in \mathbb{N})$$

and is inhabited by the above computation, denoted $\texttt{mod}(F, \alpha)$

Continuity – Further Additional Components

Further Additional Components

- a function: update $\in \mathcal{W} \rightarrow \mathcal{N} \rightarrow \mathcal{C} \rightarrow \mathcal{W}$
- namefree(t) states that t does not contain choices

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Syntax

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\begin{aligned}
t \in \texttt{Term} \; ::= \cdots & \boxed{\mid \boldsymbol{\nu} x.t} \boxed{\mid \texttt{choose}(t_1, t_2)} \\
& \mid \texttt{if } t_1 \texttt{ < } t_2 \texttt{ then } t_3 \texttt{ else } t_4 \mid t_1 + t_2 \\
T \in \texttt{Type} ::= \cdots & \boxed{\mid \texttt{pure}} \mid t_1 \cap t_2 \mid \downharpoonleft t
\end{aligned}
$$

Continuity – Further Additional Components

Further Additional Components

- a function: $\mathsf{update} \in \mathscr{W} \to \mathscr{N} \to \mathscr{C} \to \mathscr{W}$
- $\mathtt{namefree}(t)$ states that t does not contain choices

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$$
\begin{array}{rcl}
t \in \mathtt{Term} & ::= & \cdots \fbox{$\mid \boldsymbol{\nu} x.t$} \fbox{$\mid \mathtt{choose}(t_1, t_2)$} \\
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\end{array}
$$

Operational Semantics

$$w, \mathsf{update}(w, \delta, t) \vdash \mathtt{choose}(\delta, t) \mapsto \star$$

Continuity – Proof Steps

Step 1 (The Modulus is a Number)

If $\mathtt{namefree}(F)$, $\mathtt{namefree}(\alpha)$, $w \vDash F{\in}\mathbb{N}^{\mathscr{B}}$, *and* $w \vDash \alpha{\in}\mathscr{B}$, *for some world* w, *then* $w \vDash \mathtt{mod}(F,\alpha){\in}\mathbb{N}$

Continuity – Proof Steps

Step 1 (The Modulus is a Number)

If $\texttt{namefree}(F)$, $\texttt{namefree}(\alpha)$, $w \vDash F \in \mathbb{N}^{\mathscr{B}}$, *and* $w \vDash \alpha \in \mathscr{B}$, *for some world* w, *then* $w \vDash \texttt{mod}(F,\alpha) \in \mathbb{N}$

Step 2 (The Modulus is the Highest Number)

If $w, w' \vdash \texttt{mod}(F,\alpha) \longmapsto^{*} \underline{n}$ *such that* $\texttt{mod}(F,\alpha)$ *generates a fresh name* δ, *then for any world* w_0 *occurring along this computation, it must be that* $\text{choice}(w_0,\delta) \leq \text{choice}(w',\delta)$.

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Step 3 (The Modulus is the Modulus)

If $w \vDash \alpha {\equiv} \beta \in \mathscr{B}_n$ *then* $w \vDash F(\alpha) {\equiv} F(\beta) \in \mathbb{N}$.

Summary

$TT_{\mathscr{C}}^{\square}$: a type theory to program with effects

$\square \in \{Kripke, Beth, Open\}$
$\mathscr{C} \in \{Ref, CS\}$

Simple reference-based computation of continuity

What about impure functions?

Questions?