Realisability methods of proof and semantics with application to expansion
First Year Examination

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History

- Existence of paradoxes.
- Functions can be applied to any function.
- Formalization of the concept of type by Russel [Rus08] to restrict application of functions.
- Formalization of Mathematics: design of logical systems.
Design of the \( \lambda \)-calculus by Church in 1932 [Chu32].

System to investigate functions.

**Syntax:** \( M \in \Lambda ::= x \mid \lambda x.M \mid M_1M_2 \)
such that \( x \in \mathcal{V} \) (term variables).

**Rule of conversion:**

\[
(\lambda x.M)N \rightarrow_{\beta} M[x := N]
\]

\[
\lambda x.Mx \rightarrow_{\eta} M \text{ where } x \notin FV(M)
\]

**Consistency:** Church-Rosser theorem

**Church’s thesis:** “Effectively calculable functions from positive integers to positive integers are just those definable in the \( \lambda \)-calculus”.

**Models:** model \( D_\infty \) of Scott; \( \lambda \)-models of Hindley and Longo.
Typed \( \lambda \)-calculus

- Simply typed \( \lambda \)-calculus.

\[ \sigma \in \text{Type} ::= \alpha \mid \sigma_1 \rightarrow \sigma_2 \]

where \( (\alpha \in \mathcal{A}) \) is a set of type variables.

- Polymorphism: intersection types.

\[ \sigma \in \text{Type} ::= \alpha \mid \sigma_1 \rightarrow \sigma_2 \mid \sigma_1 \cap \sigma_2 \]

- \( \lambda \)-cube of Barendregt: 8 type systems with \( \neq \) expressiveness.
Example of a type systems: $\lambda \cap \Omega$ and $D$

<table>
<thead>
<tr>
<th>Axiom</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>(ref)</td>
<td>$\sigma \leq \sigma$</td>
</tr>
<tr>
<td>(tr)</td>
<td>$\sigma \leq \tau \land \tau \leq \rho \Rightarrow \sigma \leq \rho$</td>
</tr>
<tr>
<td>(inL)</td>
<td>$\sigma \cap \tau \leq \sigma$</td>
</tr>
<tr>
<td>(inR)</td>
<td>$\sigma \cap \tau \leq \tau$</td>
</tr>
<tr>
<td>(→ -∩)</td>
<td>$(\sigma \rightarrow \tau) \cap (\sigma \rightarrow \rho) \leq \sigma \rightarrow (\tau \cap \rho)$</td>
</tr>
<tr>
<td>(mon')</td>
<td>$\sigma \leq \tau \land \sigma \leq \rho \Rightarrow \sigma \leq \tau \cap \rho$</td>
</tr>
<tr>
<td>(mon)</td>
<td>$\sigma \leq \sigma' \land \tau \leq \tau' \Rightarrow \sigma \cap \tau \leq \sigma' \cap \tau'$</td>
</tr>
<tr>
<td>(→ -η)</td>
<td>$\sigma \leq \sigma' \land \tau' \leq \tau \Rightarrow \sigma \rightarrow \tau' \leq \sigma \rightarrow \tau$</td>
</tr>
<tr>
<td>(Ω)</td>
<td>$\sigma \leq \Omega$</td>
</tr>
<tr>
<td>(Ω'-lazy)</td>
<td>$\sigma \rightarrow \Omega \leq \Omega \rightarrow \Omega$</td>
</tr>
</tbody>
</table>

**Figure:** Ordering axioms on types
Example of a type system: $\lambda\cap\Omega$ and $D$

\[ \Gamma \vdash M : \sigma : \text{typing judgment} \]
\[ (\Gamma,\sigma) : (\text{typing of } M) \]

\[ \begin{array}{l}
\Gamma, x : \sigma \vdash x : \sigma \quad (\text{ax}) \\
\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \sigma \quad (\rightarrow_E) \\
\Gamma \vdash MN : \tau \\
\hline
\Gamma \vdash M : \sigma \quad \Gamma \vdash M : \tau \quad (\cap_I) \\
\Gamma \vdash M : \sigma \cap \tau \\
\hline
\Gamma \vdash M : \sigma \quad \sigma \leq \nabla \tau \quad (\leq \nabla) \\
\Gamma \vdash M : \tau \\
\end{array} \]

Figure: Typing rules
Example of a type system: $\lambda \cap \Omega$ and $D$

Type system $\lambda \cap \Omega$

- $\sigma \in \text{Type}^{\lambda \cap \Omega} ::= \alpha | \sigma_1 \rightarrow \sigma_2 | \sigma_1 \cap \sigma_2 | \Omega$.

- $B^{\lambda \cap \Omega} = \{ \Gamma = \{ x : \sigma \mid x \in V, \sigma \in \text{Type}^{\lambda \cap \Omega} \} \mid \forall x : \sigma, y : \tau \in \Gamma, \text{if } \sigma \neq \tau \text{ then } x \neq y \}$.

- $\nabla = \{ (\text{ref}), (\text{tr}), (\text{in}_L), (\text{in}_R), (\rightarrow - \cap), (\text{mon'}) , (\text{mon}), (\rightarrow - \eta), (\Omega), (\Omega' - \text{lazy}) \}$.

- The relation $\leq^{\nabla}$ is defined on types $\text{Type}^{\lambda \cap \Omega}$ and the set of axioms $\nabla$. The equivalence relation is defined by:

  $\sigma \sim^{\nabla} \tau \iff \sigma \leq^{\nabla} \tau \land \tau \leq^{\nabla} \sigma$.

- $\lambda \cap \Omega = \langle \Lambda, \text{Type}^{\lambda \cap \Omega}, \vdash \rangle$ such that $\vdash$ is type derivability relation on $B^{\lambda \cap \Omega}$, $\Lambda$ and $\text{Type}^{\lambda \cap \Omega}$ generated using the typing rules of Figure 2.
Example of a type system: $\lambda \cap \Omega$ and $D$

Type system $D$

- $\sigma \in \text{Type}^D ::= \alpha \mid \sigma_1 \rightarrow \sigma_2 \mid \sigma_1 \cap \sigma_2$.
- $\mathcal{B}^D = \{ \Gamma = \{ x : \sigma \mid x \in \mathcal{V}, \sigma \in \text{Type}^D \} \mid \forall x : \sigma, y : \tau \in \Gamma, \text{ if } \sigma \neq \tau \text{ then } x \neq y \}$.
- $\nabla = \{ (\text{ref}), (\text{tr}), (\text{in}_L), (\text{in}_R) \}$.
- The relation $\leq^\nabla$ is defined on types $\text{Type}^D$ and the set of axioms $\nabla$. The equivalence relation is defined by:
  $\sigma \sim^\nabla \tau \iff \sigma \leq^\nabla \tau \land \tau \leq^\nabla \sigma$.
- $\lambda \cap \Omega = \langle \Lambda, \text{Type}^D, \vdash \rangle$ such that $\vdash$ is type derivability relation on $\mathcal{B}^D$, $\Lambda$ and $\text{Type}^D$ generated using the typing rules of Figure 2 except $(\Omega)$. 
Properties of the $\lambda$-calculus

- **Church-Rosser** property:
  - $R$: binary relation on $\Lambda$.
  - $R$ is Church-Rosser iff $MRM_1 \land MRM_2 \Rightarrow \exists M_3, M_1 RM_3 \land M_2 RM_3$.
    - method of parallel reductions
    - method of finiteness of developments
  
  Let $L$ be a set of terms and $\rightarrow$ a reduction relation:

  $CR_L = \{ M \in L \mid M \rightarrow M_1 \land M \rightarrow M_2 \Rightarrow \exists M_3, M_1 \rightarrow M_3 \land M_2 \rightarrow M_3 \}$

  $CR^L = CR^\to_{\beta}$ and $CR \to = CR^\Lambda$.

- **Normalization** properties.
  - Standardization
  - Developments

  $SN = \{ M \in \Lambda \mid \text{each } \beta\text{-reduction from } M \text{ is finite} \}$.
  $WN = \{ M \in \Lambda \mid \exists a \beta\text{-reduction from } M \text{ which is finite} \}$.

  Goal: generalization and simplification of proof methods.
Properties of Type Systems

Properties that we might want to be held by a type system:

- Decidability of the type inference.
- Decidability of the type checking.
- Principal typing.
- Subject Reduction/Expansion.
- Strong Normalization.
Proof Methods

There exists different methods to prove properties of the $\lambda$-calculus or of type systems but not an universal one.

- $\not= \text{properties} \Rightarrow \not= \text{proof methods}.$
- Changes of framework $\Rightarrow$ all the proofs need to be reproved.
- A method may work in a framework but not in another one $\Rightarrow$ Introduction of new methods, new concept.
- **Expansion**: new concept to calculate typing from a principal one in intersection type systems
- **Reducibility**: general proof methods to prove properties of the $\lambda$-calculus.
Expansion

- Calculate a type of a term from its principal one, in a intersection type system, need more than substitution [CDCV80].
- Introduction of the mechanism of expansion.
- Development of the mechanism of expansion [KW99]: expansion variables.
- Goal: find a semantics of an intersection type system with expansion mechanism.
Reducibility

Let $\mathcal{P} \subseteq \Lambda$.

The reducibility method is based on realisability semantics.

- Interpretation of types by sets of terms such that they (mostly) turn to be subsets of $\mathcal{P}$.
- Proof of a soundness result.
Semantics of Type Systems

- Interpretation of the logical contents of a type system (Curry-Howard isomorphism).
- Study and characterization of legal types.
- Verification of the intended behavior of a type system.

A semantics is **complete** w.r.t. a type system if: a typing judgment is true in the semantics if and only if it is derivable in the type system [Hin83] (soundness: if direction).

**Realisability** semantics: types are interpreted by realizers (functions).
Reducibility in [GL02]

Let $\mathcal{P} \subseteq \Lambda$.

Definition of an interpretation of the types in $\text{Type}^{\lambda \cap \Omega}$:

- $\llbracket \alpha \rrbracket = \mathcal{P}$,
- $\llbracket \sigma \to \tau \rrbracket = \{ M \in \mathcal{P} \mid \forall N \in \llbracket \sigma \rrbracket, MN \in \llbracket \tau \rrbracket \}$,
- $\llbracket \sigma \cap \tau \rrbracket = \llbracket \sigma \rrbracket \cap \llbracket \tau \rrbracket$,
- $\llbracket \Omega \rrbracket = \Lambda$

Definition of closure properties:

- $\text{VAR}(\mathcal{P}, \mathcal{X})$: $\forall x \in \mathcal{V}, \forall n \geq 0, \forall M_1, \ldots, M_n \in \mathcal{P}, xM_1 \ldots M_n \in \mathcal{X}$.
- $\text{SAT}(\mathcal{P}, \mathcal{X})$: $\forall M, N \in \Lambda, \forall n \geq 0, \forall M_1, \ldots, M_n \in \mathcal{P}$, $M[x := N]M_1 \ldots M_n \in \mathcal{X}$ $\Rightarrow$ $(\lambda x.M)NM_1 \ldots M_n \in \mathcal{X}$.
- $\text{CLO}(\mathcal{P}, \mathcal{X})$: $\forall M \in \mathcal{X}, \lambda x.M \in \mathcal{P}$.
- $\text{INV}(\mathcal{P})$: $\forall M \in \Lambda, M \in \mathcal{P}$ $\iff$ $\lambda x.M \in \mathcal{P}$. 
Reducibility in [GL02]

- Proof of a soundness result:
  \[
  (\forall \sigma \in \text{Type}^{\lambda \cap \Omega}, \text{VAR}(\mathcal{P}, \llbracket \sigma \rrbracket) \land \text{SAT}(\mathcal{P}, \llbracket \sigma \rrbracket) \land \text{CLO}(\mathcal{P}, \llbracket \sigma \rrbracket)) \Rightarrow \\
  (\forall \sigma \in \text{Type}^{\lambda \cap \Omega}, \Gamma \vdash M : \sigma \land \sigma \not\triangleright \nabla \Omega \Rightarrow M \in \llbracket \sigma \rrbracket \subseteq \mathcal{P}).
  \]

- “Generalization” of hypotheses:
  \[
  (\text{VAR}(\mathcal{P}, \mathcal{P}) \land \text{SAT}(\mathcal{P}, \mathcal{P}) \land \text{CLO}(\mathcal{P}, \mathcal{P})) \Rightarrow (\forall \sigma \in \text{Type}^{\lambda \cap \Omega}, \Gamma \vdash M : \\
  \sigma \land \sigma \not\triangleright \nabla \Omega \land (\forall \tau \in \text{Type}, \tau \not\triangleright \nabla \Omega \Rightarrow \sigma \not\triangleright \nabla \Omega \rightarrow \tau) \Rightarrow M \in \mathcal{P}).
  \]

- Proof of the proof method:
  \[
  \text{VAR}(\mathcal{P}, \mathcal{P}) \land \text{SAT}(\mathcal{P}, \mathcal{P}) \land \text{INV}(\mathcal{P}) \Rightarrow \mathcal{P} = \Lambda.
  \]

- Proof of one of the principal result: VAR(CR, CR), SAT(CR, CR) and INV(CR) are true. Hence, \(\Lambda = \text{CR}\).
We establish that reducibility in [GL02] fails

Our counter example that [GL02] fails:
- VAR(WN, WN), SAT(WN, WN) and INV(WN) are satisfied,
- but Λ \neq WN.

Our solution to repair [GL02]:
- \( \sigma^+ \in \text{Type}^{\lambda \cap \Omega} = \{ \sigma \in \text{Type}^{\lambda \cap \Omega} \mid \sigma \sim^{\nabla} \Omega \} \).
- \( \sigma^- \in \text{Type}^{\lambda \cap \Omega^-} = \{ \sigma \in \text{Type}^{\lambda \cap \Omega} \mid \sigma \not\sim^{\nabla} \Omega \} \).
- \( \sigma \in S_1 ::= \alpha \mid \sigma_1^+ \rightarrow \sigma_2^+ \mid \sigma^- \rightarrow \sigma \mid \sigma_1 \cap \sigma_2 \).

- \( (\text{VAR}(\mathcal{P}, \mathcal{P}) \land \text{SAT}(\mathcal{P}, \mathcal{P}) \land \text{CLO}(\mathcal{P}, \mathcal{P})) \Rightarrow (\forall \sigma \in \text{Type}^{\lambda \cap \Omega}, \Gamma \vdash M : \sigma \land \sigma \not\sim^{\nabla} \Omega \land (\sigma = \tau \rightarrow \rho \Rightarrow \tau, \rho \in S_1 \land \rho \not\sim^{\nabla} \Omega) \Rightarrow M \in \mathcal{P}) \).
Definition of an interpretation of the types in $\text{Type}^D$:

1. $\llbracket \alpha \rrbracket = \text{CR},$
2. $\llbracket \sigma \rightarrow \tau \rrbracket = \{ M \in \text{CR} \mid \forall N \in \llbracket \sigma \rrbracket, MN \in \llbracket \tau \rrbracket \},$
3. $\llbracket \sigma \cap \tau \rrbracket = \llbracket \sigma \rrbracket \cap \llbracket \tau \rrbracket.$

Proof of a soundness result:
$\forall \sigma \in \text{Type}^D, \Gamma \vdash M : \sigma \Rightarrow M \in \llbracket \sigma \rrbracket \subseteq \text{CR}.$
Confluence of developments:

- Let $x \in \mathcal{V}' = \mathcal{V} \setminus \{c\}$, then
  \[ M \in \Lambda_c ::= x \mid \lambda x. M \mid cM_1M_2 \mid (\lambda x. M_1)M_2 \]
- $\forall M \in \Lambda_c$, $M$ is typable in the system $D$.
- A one step development of $(M, \mathcal{F})$ is a pair $(M', \mathcal{F}')$ such that
  $M \rightarrow_{\beta} M'$, $\mathcal{F}$ is a set of redexes in $M$ and $\mathcal{F}'$ is the set of residuals
  of $\mathcal{F}$ in $M'$.
- Development = reflexive and transitive closure of a one step development.
- Proof of the confluence of developments using the confluence of $\Lambda_c$.

Parallel reduction:

- $M \rightarrow_{1} M' \iff \exists \mathcal{F}, \mathcal{F}', (M, \mathcal{F}) \rightarrow_{\beta} (M', \mathcal{F}')$.
- $\rightarrow^* = \rightarrow_{1}^*$.
- $\Lambda = CR$. 
Our extension of reducibility of [KS07]

We have:

- Adapted the method to $\beta I$: $\Lambda_I = CR^{\Lambda_I}$.
- Extended and generalized the method to $\beta \eta$: $\Lambda = CR_{\beta \eta}$.
- Formalized the various steps of the proof so it can be further extended.
Semantics of an intersection type system with expansion

Syntax:
- a set of expansion variables $e \in E\text{-var}$.
- a rule for introduction of $e$:

$$\frac{\Gamma \vdash M : \sigma}{e\Gamma \vdash M : e\sigma} e_I$$

Goal: Find a complete semantics for an intersection type system with expansion mechanism.

Intermediate Goal: Find a complete semantics for an intersection type system with expansion variables.
Semantics of an intersection type system with expansion

We have 2 results so far:

- [KNRW06] Complete semantics for an intersection type system
  - with only one expansion variable
  - without the expansion mechanism

- [KNRW07] Complete semantics for an intersection type system
  - with an infinity of expansion variables
  - without the expansion mechanism

Use of the λ-calculus with some decorations.
Future work

- Find a complete semantics for an intersection type system with the expansion mechanism. (September 2007 – September 2008).
  - Realisability semantics
  - Denotational semantics
  - Operational semantics
- Generalization of proof methods of properties of the $\lambda$-calculus using reducibility. (September 2008 – July 2009).
  - In the framework of the $\lambda$-cube
  - In the framework of expansion
- Generalization and extension of PTSs. (September 2007 – July 2009).
  - Capture of the Strong Normalization property.
  - Extension with: Unified binder, Type inclusion, $\Pi$-application and abbreviations.
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