

Reducibility proofs in the λ -calculus

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Abstract. Reducibility has been used to prove a number of properties of the λ -calculus and is well known to offer on one hand very general proofs which can be applied to a number of instantiations, and on the other hand, to be quite mysterious and inflexible. In this paper, we look at two related but different results in λ -calculi with intersection types.

1. We show that one such result (which aims at giving reducibility proofs of Church-Rosser, standardisation and weak normalisation for the untyped λ -calculus) faces serious problems which break the reducibility method and then we provide a proposal to partially repair the method.
2. Then, we consider a second result whose purpose is to use reducibility for typed terms in order to show the Church-Rosser of β -developments for the untyped terms (and hence the Church-Rosser of β -reduction). In this second result, strong normalisation is not needed. We extend the second result to encompass both βI - and $\beta\eta$ -reduction rather than simply β -reduction.

Keywords: Lambda-Calculus, Reducibility, Church-Rosser, Developments

1. Introduction

Based on realisability semantics [Kle45], the reducibility method has been developed by Tait [Tai67] in order to prove the normalisation of some functional theories. The idea is to interpret types by sets of λ -terms closed under some properties. Krivine [Kri90] uses reducibility to prove the strong normalisation of system \mathcal{D} . Koletsos [Kol85] proves that the set of simply typed λ -terms has the Church-Rosser property. Gallier [Gal97, Gal03] uses some aspects of Koletsos's method to prove a number of results such as the strong normalisation of the λ -terms that are typable in systems like \mathcal{D} or $\mathcal{D}\Omega$ [Kri90]. In particular, Gallier states some conditions a property needs to satisfy in order to be enjoyed by some typable terms under some restrictions. Similarly, Ghilezan and Likavec [GL02] state some conditions a property on λ -terms has to satisfy in order to be held by all λ -terms that are typable under some restriction on

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types in a type system which is close to $\mathcal{D}\Omega$. Additionally Ghilezan and Likavec state a condition that a property needs to satisfy in order to step from the statement “a λ -term typable under some restrictions on types holds the property” to the statement “a λ -term of the untyped λ -calculus holds the property”. If successful, the method designed by Ghilezan and Likavec would provide an attractive method for establishing properties like Church-Rosser for all the untyped λ -terms, simply by showing easier conditions on typed terms. However, we show in this paper that Ghilezan and Likavec’s method fails for the typed terms, and that also the step of passing from typed to untyped terms fails. We outline the obstacle we faced when trying to repair the first result and explain how far we have been able to go in our attempt to repair this first result (we reach a similar result to one already obtained by Ghilezan and Likavec). The second result seems unrepairable. Ghilezan and Likavec also present a weaker version of their method for a type system similar to system \mathcal{D} , which allows using reducibility to prove properties of the terms typable by this system, namely the strongly normalisable terms. As far as we know, this portion of their result is correct. (They do not actually apply this weaker method to any sets of terms.)

In addition to the method proposed by Ghilezan and Likavec (which does not actually work for the full untyped λ -calculus), other steps of establishing properties like Church-Rosser (also called confluence) for typed λ -terms and concluding the properties for all the untyped λ -terms have been successfully exploited in the literature. Koletsos and Stavrinou [KS08] use reducibility to state that the λ -terms that are typable in system \mathcal{D} hold the Church-Rosser property. Using this result together with a method based on β -developments [Klo80, Kri90], they show that β -developments are Church-Rosser and this in turn will imply the confluence of the untyped λ -calculus. Although Klop proves the confluence of β -developments [BBKV76], his proof is based on strong normalisation whereas the Koletsos and Stavrinou’s proof only uses an embedding of β -developments in the reduction of typable λ -terms. In this paper, we apply Koletsos and Stavrinou’s method to βI -reduction and then generalise it to $\beta\eta$ -reduction.

In section 2 we introduce the formal machinery and establish the basic needed lemmas. In section 3 we present the reducibility method used by Ghilezan and Likavec and show that it fails at a number of important propositions which makes it inapplicable to the full untyped λ -calculus, although a version of their method works for the strongly normalisable terms. We give counterexamples which show that all the conditions stated in Ghilezan and Likavec’s paper are satisfied, yet the claimed property does not hold. In section 4 we give some indications on the limits of the method. We show how these limits affect the salvation of the method, we partially salvage it and we show that this can now be correctly used to establish confluence, standardisation and weak head normal forms but only for restricted sets of lambda terms and types (that we believe to be equal to the set of strongly normalisable terms). We also point out some links between the work done by Ghilezan and Likavec and the work done by Gallier. In section 5, we introduce the reducibility semantics for both βI - and $\beta\eta$ -reduction and establish its soundness. Then, we show that all typable terms satisfy the Church-Rosser property. In section 6 we adapt the Church-Rosser proof of Koletsos and Stavrinou [KS08] to βI -reduction. In section 7 we non-trivially generalise Koletsos and Stavrinou’s method to handle $\beta\eta$ -reduction. We formalise $\beta\eta$ -residuals and $\beta\eta$ -developments in section 7.1. Then, we compare our notion of $\beta\eta$ -residuals with those of Curry and Feys [CF58] and Klop [Klo80] in section 7.2, establishing that we allow less residuals than Klop but we believe more residuals than Curry and Feys. Finally, we establish in section 7.3 the confluence of $\beta\eta$ -developments and hence of $\beta\eta$ -reduction. We conclude in section 8. Due to space limitations, proofs are mostly omitted but can be downloaded from the author’s web pages (in particular, from <http://www.macs.hw.ac.uk/~fairouz/papers/drafts/long-fund-inf-cr.pdf>).

2. The Formal Machinery

In this section we provide some known formal machinery and introduce new definitions and lemmas that are necessary for the paper. Let n, m be metavariables which range over the set of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$. We take as convention that if a metavariable v ranges over a set s then the metavariables v_i such that $i \geq 0$ and the metavariables $v', v'', \text{etc.}$ also range over s .

A binary relation is a set of pairs. Let rel range over binary relations. Let $\text{dom}(rel) = \{x \mid \langle x, y \rangle \in rel\}$ and $\text{ran}(rel) = \{y \mid \langle x, y \rangle \in rel\}$. A function is a binary relation fun such that if $\{\langle x, y \rangle, \langle x, z \rangle\} \subseteq fun$ then $y = z$. Let fun range over functions. Let $s \rightarrow s' = \{fun \mid \text{dom}(fun) \subseteq s \wedge \text{ran}(fun) \subseteq s'\}$.

Given n sets s_1, \dots, s_n , where $n \geq 2$, $s_1 \times \dots \times s_n$ stands for the set of all the tuples built on the sets s_1, \dots, s_n . If $x \in s_1 \times \dots \times s_n$, then $x = \langle x_1, \dots, x_n \rangle$ such that $x_i \in s_i$ for all $i \in \{1, \dots, n\}$.

2.1. Familiar background on λ -calculus

This section consists of one long definition of some familiar (mostly standard) concepts of the λ -calculus and one lemma which deals with the shape of reductions.

Definition 2.1. 1. let $x, y, z, \text{etc.}$ range over \mathcal{V} , a countable infinite set of λ -term variables. The set of terms of the λ -calculus is defined as follows:

$$M \in \Lambda ::= x \mid (\lambda x.M) \mid (M_1 M_2)$$

We let $M, N, P, Q, \text{etc.}$ range over Λ . We assume the usual definition of subterms: we write $N \subseteq M$ if N is a subterm of M . We also assume the usual convention for parenthesis and omit these when no confusion arises. In particular, we write $M N_1 \dots N_n$ instead of $(\dots (M N_1) N_2 \dots N_{n-1}) N_n$.

We take terms modulo α -conversion and use the Barendregt convention (BC) where the names of bound variables differ from the free ones. When two terms M and N are equal (modulo α), we write $M = N$. We write $\text{fv}(M)$ for the set of the free variables of term M .

2. Let $n \geq 0$. We define $M^n(N)$, by induction on n , as follows: $M^0(N) = N$ and $M^{n+1}(N) = M(M^n(N))$.

3. The set of paths is defined as follows:

$$p \in \text{Path} ::= 0 \mid 1.p \mid 2.p$$

We define $M|_p$ by: $M|_0 = M$, $(\lambda x.M)|_{1.p} = M|_p$, $(MN)|_{1.p} = M|_p$ and $(MN)|_{2.p} = N|_p$. We define $2^n.p$ by induction on $n \geq 0$: $2^0.p = p$ and $2^{n+1}.p = 2^n.2.p$.

4. The set $\Lambda\text{I} \subset \Lambda$, of terms of the λI -calculus is defined by the following rules:

- (a) If $x \in \mathcal{V}$ then $x \in \Lambda\text{I}$.
- (b) If $x \in \text{fv}(M)$ and $M \in \Lambda\text{I}$ then $\lambda x.M \in \Lambda\text{I}$.
- (c) If $M, N \in \Lambda\text{I}$ then $MN \in \Lambda\text{I}$.

5. We define as usual the substitution $M[x := N]$ of N for all free occurrences of x in M . We let $M[x_i := N_i, \dots, x_n := N_n]$ be the simultaneous substitution of N_i for all free occurrences of x_i in M for $1 \leq i \leq n$.

6. We define the following four common relations:

- **Beta** ::= $\langle (\lambda x.M)N, M[x := N] \rangle$.
- **Betal** ::= $\langle (\lambda x.M)N, M[x := N] \rangle$, where $x \in \text{fv}(M)$.
- **Eta** ::= $\langle \lambda x.Mx, M \rangle$, where $x \notin \text{fv}(M)$.
- **BetaEta** = **Beta** \cup **Eta**.

Let $\langle r, s \rangle \in \{ \langle \mathbf{Beta}, \beta \rangle, \langle \mathbf{Betal}, \beta I \rangle, \langle \mathbf{Eta}, \eta \rangle, \langle \mathbf{BetaEta}, \beta \eta \rangle \}$. We define \mathcal{R}^s to be $\{ L \mid \langle L, R \rangle \in r \}$. If $\langle L, R \rangle \in r$ then we call L a s -redex and R the s -contractum of L (or the L s -contractum).

We define the ternary relation \rightarrow_s as follows:

- $M \xrightarrow{0}_s M'$ if $\langle M, M' \rangle \in r$.
- $\lambda x.M \xrightarrow{1.p}_s \lambda x.M'$ if $M \xrightarrow{p}_s M'$.
- $MN \xrightarrow{1.p}_s M'N$ if $M \xrightarrow{p}_s M'$.
- $NM \xrightarrow{2.p}_s NM'$ if $M \xrightarrow{p}_s M'$.

We define the binary relation \rightarrow_s (we use the same name as for the just defined ternary relation \rightarrow_s to simplify the notations) as follows: $M \rightarrow_s M'$ if there exists p such that $M \xrightarrow{p}_s M'$. We define $\mathcal{R}_M^s = \{ p \mid M|_p \in \mathcal{R}^s \}$.

7. Let $M \in \Lambda$ and $\mathcal{F} \subseteq \Lambda$. $\mathcal{F} \upharpoonright M = \{ N \mid N \in \mathcal{F} \wedge N \subseteq M \}$.

8. $\rightarrow_{h\beta} ::= \langle \lambda x_1 \dots x_n. (\lambda x.M_0)M_1 \dots M_m, \lambda x_1 \dots x_n. M_0[x := M_1]M_2 \dots M_m \rangle$, where $n \geq 0$ and $m \geq 1$.

If $\langle L, R \rangle \in \rightarrow_{h\beta}$ then $L = \lambda x_1 \dots x_n. (\lambda x.M_0)M_1 \dots M_m$ where $n \geq 0$ and $m \geq 1$ and $(\lambda x.M_0)M_1$ is called the β -head redex of L .

We define the binary relation $\rightarrow_{i\beta}$ as $\rightarrow_\beta \setminus \rightarrow_{h\beta}$.

9. Let $r \in \{ \rightarrow_\beta, \rightarrow_\eta, \rightarrow_{\beta\eta}, \rightarrow_{\beta I}, \rightarrow_{h\beta}, \rightarrow_{i\beta} \}$. We use \rightarrow_r^* to denote the reflexive transitive closure of \rightarrow_r . We let \simeq_r denote the equivalence relation induced by \rightarrow_r . If the r -reduction from M to N is in k steps, we write $M \rightarrow_r^k N$.

10. Let $r \in \{ \beta I, \beta \eta \}$ and $n \geq 0$. A term $(\lambda x.M')N'_0 N'_1 \dots N'_n$ is a direct r -reduct of a term $(\lambda x.M)N_0 N_1 \dots N_n$ iff $M \rightarrow_r^* M'$ and $\forall i \in \{0, \dots, n\}$. $N_i \rightarrow_r^* N'_i$.

11. $\mathbf{NF}_\beta = \{ \lambda x_1 \dots \lambda x_n. x_0 N_1 \dots N_m \mid n, m \geq 0, N_1, \dots, N_m \in \mathbf{NF}_\beta \}$.

12. $\mathbf{WN}_\beta = \{ M \in \Lambda \mid \exists N \in \mathbf{NF}_\beta, M \rightarrow_\beta^* N \}$.

13. Let $r \in \{ \beta, \beta I, \beta \eta \}$.

- We say that M has the Church-Rosser property for r (has r -CR) if whenever $M \rightarrow_r^* M_1$ and $M \rightarrow_r^* M_2$ then there is an M_3 such that $M_1 \rightarrow_r^* M_3$ and $M_2 \rightarrow_r^* M_3$.
 - $\text{CR}^r = \{M \mid M \text{ has } r\text{-CR}\}$.
 - $\text{CR}_0^r = \{xM_1 \dots M_n \mid n \geq 0 \wedge x \in \mathcal{V} \wedge (\forall i \in \{1, \dots, n\}, M_i \in \text{CR}^r)\}$.
 - We use CR to denote CR^β and CR_0 to denote CR_0^β .
 - A term is a weak head normal form if it is a λ -abstraction (a term of the form $\lambda x.M$) or if it starts with a variable (a term of the form $xM_1 \dots M_n$). A term is weakly head normalising if it reduces to a weak head normal form. Let $\mathbf{W}^r = \{M \in \Lambda \mid \exists n \geq 0, \exists x \in \mathcal{V}, \exists P, P_1, \dots, P_n \in \Lambda, M \rightarrow_r^* \lambda x.P \text{ or } M \rightarrow_r^* xP_1 \dots P_n\}$. We use \mathbf{W} to denote \mathbf{W}^β .
14. We say that M has the standardisation property if whenever $M \rightarrow_\beta^* N$ then there is an M' such that $M \rightarrow_h^* M'$ and $M' \rightarrow_i^* N$. Let $\mathbf{S} = \{M \in \Lambda \mid M \text{ has the standardisation property}\}$.

The next lemma deals with the shape of reductions.

Lemma 2.1.

1. $M \xrightarrow{p}_{\beta\eta} M'$ iff $(M \xrightarrow{p}_\beta M' \text{ or } M \xrightarrow{p}_\eta M')$.
2. If $x \in \text{fv}(M_1)$ then $\text{fv}((\lambda x.M_1)M_2) = \text{fv}(M_1[x := M_2])$ and if $(\lambda x.M_1)M_2 \in \Lambda\text{I}$ then $M_1[x := M_2] \in \Lambda\text{I}$.
3. If $M \rightarrow_{\beta\eta}^* M'$ then $\text{fv}(M') \subseteq \text{fv}(M)$.
4. If $M \rightarrow_{\beta\text{I}}^* M'$ then $\text{fv}(M) = \text{fv}(M')$ and if $M \in \Lambda\text{I}$ then $M' \in \Lambda\text{I}$.
5. $\lambda x.M \xrightarrow{p}_{\beta\eta} P$ iff either $(p = 1.p', P = \lambda x.M'$ and $M \xrightarrow{p'}_{\beta\eta} M')$ or $(p = 0, M = Px$ and $x \notin \text{fv}(P))$.
6. Let $r \in \{\beta\text{I}, \beta\eta\}$, $n \geq 0$, P is not a direct r -reduct of $(\lambda x.M)N_0 \dots N_n$ and $(\lambda x.M)N_0 \dots N_n \rightarrow_r^k P$. Then the following holds:
 - (a) $k \geq 1$, and if $k = 1$ then $P = M[x := N_0]N_1 \dots N_n$.
 - (b) There exists a direct r -reduct $(\lambda x.M')N'_0 N'_1 \dots N'_n$ of $(\lambda x.M)N_0 \dots N_n$ such that $M'[x := N'_0]N'_1 \dots N'_n \rightarrow_r^* P$.
7. Let $r \in \{\beta\text{I}, \beta\eta\}$, $n \geq 0$ and $(\lambda x.M)N_0 N_1 \dots N_n \rightarrow_r^* P$. There exists P' such that $P \rightarrow_r^* P'$ and if $(r = \beta\text{I} \text{ and } x \in \text{fv}(M))$ or $r = \beta\eta$ then $M[x := N_0]N_1 \dots N_n \rightarrow_r^* P'$.
8. There exists M' such that $M \xrightarrow{p}_r M'$ iff $p \in \mathcal{R}_M^r$.
9. If $M \xrightarrow{p}_r M_1$ and $M \xrightarrow{p}_r M_2$ then $M_1 = M_2$

2.2. Formalising the background on developments

In this section we go through some needed background from [Kri90] on developments and we precisely formalise and establish all the necessary properties. Throughout the paper, we take c to be a metavariable ranging over \mathcal{V} . As far as we know, this is the first precise formalisation of developments.

The next definition adapts Λ_c of [Kri90] to deal with βI - and $\beta\eta$ -reduction. Basically, ΛI_c is Λ_c where in the abstraction construction rule (R1).2, we restrict abstraction to ΛI . In $\Lambda\eta_c$ we introduce the new rule (R4) and replace the abstraction rule of Λ_c by (R1).3 and (R1).4.

Definition 2.2. ($\Lambda\eta_c, \Lambda I_c$)

1. We let \mathcal{M}_c range over $\Lambda\eta_c, \Lambda I_c$ defined as follows (note that $\Lambda I_c \subset \Lambda I$):

(R1) If x is a variable distinct from c then

1. $x \in \mathcal{M}_c$.
2. If $M \in \Lambda I_c$ and $x \in \text{fv}(M)$ then $\lambda x.M \in \Lambda I_c$.
3. If $M \in \Lambda\eta_c$ then $\lambda x.M[x := c(cx)] \in \Lambda\eta_c$.
4. If $Nx \in \Lambda\eta_c$ such that $x \notin \text{fv}(N)$ and $N \neq c$ then $\lambda x.Nx \in \Lambda\eta_c$.

(R2) If $M, N \in \mathcal{M}_c$ then $cMN \in \mathcal{M}_c$.

(R3) If $M, N \in \mathcal{M}_c$ and M is a λ -abstraction then $MN \in \mathcal{M}_c$.

(R4) If $M \in \Lambda\eta_c$ then $cM \in \Lambda\eta_c$.

Here is a lemma related to terms of \mathcal{M}_c .

Lemma 2.2. (Generation)

1. $M[x := c(cx)] \neq x$ and for any N , $M[x := c(cx)] \neq Nx$.
2. Let $x \notin \text{fv}(M)$. Then, $M[y := c(cx)] \neq x$ and for any N , $M[y := c(cx)] \neq Nx$.
3. If $M \in \mathcal{M}_c$ then $M \neq c$.
4. If $M, N \in \mathcal{M}_c$ then $M[x := N] \neq c$.
5. Let $MN \in \mathcal{M}_c$. Then $N \in \mathcal{M}_c$ and either
 - $M = cM'$ where $M' \in \mathcal{M}_c$ or
 - $M = c$ and $\mathcal{M}_c = \Lambda\eta_c$ or
 - $M = \lambda x.P$ is in \mathcal{M}_c
6. If $c^n(M) \in \mathcal{M}_c$ then $M \in \mathcal{M}_c$.
7. If $M \in \Lambda\eta_c$ and $n \geq 0$ then $c^n(M) \in \Lambda\eta_c$.
8. If $\lambda x.P \in \Lambda\eta_c$ then $x \neq c$ and either

- $P = Nx$ where $N, Nx \in \Lambda\eta_c$ where $x \notin \text{fv}(N)$ and $N \neq c$ or
 - $P = N[x := c(cx)]$ where $N \in \Lambda\eta_c$
9. If $\lambda x.P \in \Lambda\mathbf{I}_c$ then $x \neq c$, $x \in \text{fv}(P)$ and $P \in \Lambda\mathbf{I}_c$.
10. If $M, N \in \mathcal{M}_c$ and $x \neq c$ then $M[x := N] \in \mathcal{M}_c$.
11. Let $y \notin \{x, c\}$. Then:
- if $M[x := c(cx)] = y$ then $M = y$,
 - if $M[x := c(cx)] = Py$ then $M = Ny$ and $P = N[x := c(cx)]$,
 - if $M[x := c(cx)] = \lambda y.P$ then $M = \lambda y.N$ and $P = N[x := c(cx)]$.
 - if $M[x := c(cx)] = PQ$ then either $M = x$, $P = c$ and $Q = cx$ or $M = P'Q'$ and $P = P'[x := c(cx)]$ and $Q = Q'[x := c(cx)]$.
 - if $M[x := c(cx)] = (\lambda y.P)Q$ then $M = (\lambda y.P')Q'$ and $P = P'[x := c(cx)]$ and $Q = Q'[x := c(cx)]$.
12. Let $M \in \Lambda\eta_c$.
- (a) If $M = \lambda x.P$ then $P \in \Lambda\eta_c$.
 - (b) If $M = \lambda x.Px$ then $Px, P \in \Lambda\eta_c$, $x \notin \text{fv}(P) \cup \{c\}$ and $P \neq c$.
13. (a) Let $x \neq c$. $M[x := c(cx)] \xrightarrow{p}_{\beta\eta} M'$ iff $M' = N[x := c(cx)]$ and $M \xrightarrow{p}_{\beta\eta} N$.
- (b) Let $n \geq 0$. If $c^n(M) \xrightarrow{p}_{\beta\eta} M'$ then $p = 2^n \cdot p'$ and there exists $N \in \Lambda\eta_c$ such that $M' = c^n(N)$ and $M \xrightarrow{p'}_{\beta\eta} N$.

Here is a lemma about the set and paths of redexes in a term:

Lemma 2.3. Let $r \in \{\beta\mathbf{I}, \beta\eta\}$ and $\mathcal{F} \subseteq \mathcal{R}_M^r$.

- If $M \in \mathcal{V}$ then $\mathcal{R}_M^r = \emptyset$ and $\mathcal{F} = \emptyset$.
- If $M = \lambda x.N$ then $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}_N^r$ and:
 - if $M \in \mathcal{R}^r$ then $\mathcal{R}_M^r = \{0\} \cup \{1.p \mid p \in \mathcal{R}_N^r\}$ and $\mathcal{F} \setminus \{0\} = \{1.p \mid p \in \mathcal{F}'\}$.
 - else, $\mathcal{R}_M^r = \{1.p \mid p \in \mathcal{R}_N^r\}$ and $\mathcal{F} = \{1.p \mid p \in \mathcal{F}'\}$.
- If $M = PQ$ then $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}_P^r$, $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq \mathcal{R}_Q^r$ and:
 - if $M \in \mathcal{R}^r$ then $\mathcal{R}_M^r = \{0\} \cup \{1.p \mid p \in \mathcal{R}_P^r\} \cup \{2.p \mid p \in \mathcal{R}_Q^r\}$ and $\mathcal{F} \setminus \{0\} = \{1.p \mid p \in \mathcal{F}_1\} \cup \{2.p \mid p \in \mathcal{F}_2\}$.
 - else, $\mathcal{R}_M^r = \{1.p \mid p \in \mathcal{R}_P^r\} \cup \{2.p \mid p \in \mathcal{R}_Q^r\}$ and $\mathcal{F} = \{1.p \mid p \in \mathcal{F}_1\} \cup \{2.p \mid p \in \mathcal{F}_2\}$.

The next lemma shows the role of redexes w.r.t. substitutions involving c .

Lemma 2.4. Let $r \in \{\beta\eta, \beta\mathbf{I}\}$. and $x \neq c$.

1. $M \in \mathcal{R}^{\beta\eta}$ iff $M[x := c(cx)] \in \mathcal{R}^{\beta\eta}$.
2. If $p \in \mathcal{R}_M^{\beta\eta}$ then $M[x := c(cx)]|_p = M|_p[x := c(cx)]$.
3. $p \in \mathcal{R}_{\lambda x.M[x:=c(cx)]}^{\beta\eta}$ iff $p = 1.p'$ and $p' \in \mathcal{R}_{M[x:=c(cx)]}^{\beta\eta}$.
4. $\mathcal{R}_{M[x:=c(cx)]}^{\beta\eta} = \mathcal{R}_M^{\beta\eta}$.
5. $\mathcal{R}_{c^n(M)}^{\beta\eta} = \{2^n.p \mid p \in \mathcal{R}_M^{\beta\eta}\}$.

The next lemma shows that any element $(\lambda x.P)Q$ of ΛI_c (resp. $\Lambda \eta_c$) is a βI - (resp. $\beta \eta$ -) redex, that ΛI_c (resp. $\Lambda \eta_c$) contains all the βI -redexes (resp. $\beta \eta$ -redexes) of all its terms and generalises a lemma given in [Kri90] (and used in [KS08]) stating that $\Lambda \eta_c$ and ΛI_c are closed under $\rightarrow_{\beta \eta}$ - resp. $\rightarrow_{\beta I}$ -reduction.

Lemma 2.5. 1. Let $(\mathcal{M}_c, r) \in \{(\Lambda I_c, \beta I), (\Lambda \eta_c, \beta \eta)\}$ and $M \in \mathcal{M}_c$.

- (a) If $M = (\lambda x.P)Q$ then $M \in \mathcal{R}^r$.
- (b) If $p \in \mathcal{R}_M^r$ then $M|_p \in \mathcal{M}_c$.
2. (a) If $M \in \Lambda \eta_c$ and $M \rightarrow_{\beta \eta} M'$ then $M' \in \Lambda \eta_c$.
- (b) If $M \in \Lambda I_c$ and $M \rightarrow_{\beta I} M'$ then $M' \in \Lambda I_c$.

The next definition, taken from [Kri90], erases all the c 's from an \mathcal{M}_c -term. We extend it to paths.

Definition 2.3. $(| _ |^c)$

We define $|M|^c$ and $|\langle M, p \rangle|^c$ inductively as follows:

- $|x|^c = x$
- $|\lambda x.N|^c = \lambda x.|N|^c$, if $x \neq c$
- $|cP|^c = |P|^c$
- $|NP|^c = |N|^c|P|^c$ if $N \neq c$
- $|\langle M, 0 \rangle|^c = 0$
- $|\langle \lambda x.M, 1.p \rangle|^c = 1.|\langle M, p \rangle|^c$, if $x \neq c$
- $|\langle cM, 2.p \rangle|^c = |\langle M, p \rangle|^c$
- $|\langle NM, 2.p \rangle|^c = 2.|\langle M, p \rangle|^c$ if $N \neq c$
- $|\langle MN, 1.p \rangle|^c = 1.|\langle M, p \rangle|^c$

Let $\mathcal{F} \subseteq \text{Path}$ then we define $|\langle M, \mathcal{F} \rangle|^c = \{|\langle M, p \rangle|^c \mid p \in \mathcal{F}\}$.

Now, c^n is indeed erased from $|c^n(M)|^c$ and from $|c^n(N)|^c$ for any $c^n(N)$ subterm of M .

Lemma 2.6. 1. Let $n \geq 0$ then $|c^n(M)|^c = |M|^c$.

2. $|\langle c^n(M), \mathcal{R}_{c^n(M)}^{\beta\eta} \rangle|^c = |\langle M, \mathcal{R}_M^{\beta\eta} \rangle|^c$.
3. $|\langle c^n(M), 2^n.p \rangle|^c = |\langle M, p \rangle|^c$.
4. Let $|M|^c = P$.
 - If $P \in \mathcal{V}$ then $\exists n \geq 0$ such that $M = c^n(P)$.
 - If $P = \lambda x.Q$ then $\exists n \geq 0$ such that $M = c^n(\lambda x.N)$ and $|N|^c = Q$.

- If $P = P_1 P_2$ then $\exists n \geq 0$ such that $M = c^n(M_1 M_2)$, $M_1 \neq c$, $|M_1|^c = P_1$ and $|M_2|^c = P_2$.

Furthermore, if the c -erasure of two paths of M are equal, then these paths are also equal and inside a term, substituting x by $c(cx)$ is undone by c -erasure, c is definitely erased from the free variables of $|M|^c$, erasure propagates through substitutions and c -erasing an ΛI_c -term returns an ΛI -term.

Lemma 2.7. 1. Let $r \in \{\beta I, \beta \eta\}$. If $p, p' \in \mathcal{R}_M^r$ and $|\langle M, p \rangle|^c = |\langle M, p' \rangle|^c$ then $p = p'$.

2. Let $x \neq c$. Then, $|M[x := c(cx)]|^c = |M|^c$.

3. Let $x \neq c$ and $p \in \mathcal{R}_M^{\beta \eta}$. Then, $|\langle M[x := c(cx)], p \rangle|^c = |\langle M, p \rangle|^c$.

4. If $M \in \mathcal{M}_c$ then $\text{fv}(M) \setminus \{c\} = \text{fv}(|M|^c)$.

5. If $M, N \in \mathcal{M}_c$ and $x \neq c$ then $|M[x := N]|^c = |M|^c[x := |N|^c]$.

6. If $M \in \Lambda I_c$ then $|M|^c \in \Lambda I$.

7. Let $(\mathcal{M}_c, r) \in \{(\Lambda I_c, \beta I), (\Lambda \eta_c, \beta \eta)\}$ and $M, M_1, N_1, M_2, N_2 \in \mathcal{M}_c$.

(a) If $p \in \mathcal{R}_M^r$ and $M \xrightarrow{p}_r M'$ then $|M|^c \xrightarrow{p'}_r |M'|^c$ such that $p' = |\langle M, p \rangle|^c$.

(b) Let $x \neq c$, $|\langle M_1, \mathcal{R}_{M_1}^r \rangle|^c \subseteq |\langle M_2, \mathcal{R}_{M_2}^r \rangle|^c$, $|\langle N_1, \mathcal{R}_{N_1}^r \rangle|^c \subseteq |\langle N_2, \mathcal{R}_{N_2}^r \rangle|^c$, $|M_1|^c = |M_2|^c$ and $|N_1|^c = |N_2|^c$. Then, $|\langle M_1[x := N_1], \mathcal{R}_{M_1[x := N_1]}^r \rangle|^c \subseteq |\langle M_2[x := N_2], \mathcal{R}_{M_2[x := N_2]}^r \rangle|^c$.

(c) Let $|\langle M_1, \mathcal{R}_{M_1}^r \rangle|^c \subseteq |\langle M_2, \mathcal{R}_{M_2}^r \rangle|^c$ and $|M_1|^c = |M_2|^c$. If $M_1 \xrightarrow{p_1}_r M'_1$, $M_2 \xrightarrow{p_2}_r M'_2$ such that $|\langle M_1, p_1 \rangle|^c = |\langle M_2, p_2 \rangle|^c$ then $|\langle M'_1, \mathcal{R}_{M'_1}^r \rangle|^c \subseteq |\langle M'_2, \mathcal{R}_{M'_2}^r \rangle|^c$.

2.3. Background on Types and Type Systems

In this section we give the background necessary for the type systems used in this paper.

Definition 2.4. Let $i \in \{1, 2\}$.

1. Let \mathcal{A} be a countably infinite set of type variables, let α range over \mathcal{A} and let $\Omega \notin \mathcal{A}$ be a constant type. The sets of types $\mathbf{Type}^1 \subset \mathbf{Type}^2$ are defined as follows:

$$\sigma \in \mathbf{Type}^1 ::= \alpha \mid \sigma_1 \rightarrow \sigma_2 \mid \sigma_1 \cap \sigma_2$$

$$\tau \in \mathbf{Type}^2 ::= \alpha \mid \tau_1 \rightarrow \tau_2 \mid \tau_1 \cap \tau_2 \mid \Omega$$

2. We let $\Gamma \in \mathcal{B}^1 = \{\{x_1 : \sigma_1, \dots, x_n : \sigma_n\} \mid \forall i, j \in \{1, \dots, n\}. x_i = x_j \Rightarrow \sigma_i = \sigma_j\}$ and $\Gamma, \Delta \in \mathcal{B}^2 = \{\{x_1 : \tau_1, \dots, x_n : \tau_n\} \mid \forall i, j \in \{1, \dots, n\}. x_i = x_j \Rightarrow \tau_i = \tau_j\}$. We define $\text{dom}(\Gamma) = \{x \mid x : \sigma \in \Gamma\}$. When $\text{dom}(\Gamma_1) \cap \text{dom}(\Gamma_2) = \emptyset$, we write Γ_1, Γ_2 for $\Gamma_1 \cup \Gamma_2$. We write $\Gamma, x : \sigma$ for $\Gamma, \{x : \sigma\}$ and $x : \sigma$ for $\{x : \sigma\}$. We denote $\Gamma = x_m : \sigma_m, \dots, x_n : \sigma_n$ where $n \geq m \geq 0$, by $(x_i : \sigma_i)_n^m$. If $m = 1$, we simply denote Γ by $(x_i : \sigma_i)_n$.

If $\Gamma_1 = (x_i : \tau_i)_n, (y_i : \tau'_i)_p$ and $\Gamma_2 = (x_i : \tau'_i)_n, (z_i : \tau''_i)_q$ where x_1, \dots, x_n are the only shared variables, then $\Gamma_1 \cap \Gamma_2 = (x_i : \tau_i \cap \tau'_i)_n, (y_i : \tau''_i)_p, (z_i : \tau'_i)_q$.

Let $X \subseteq \mathcal{V}$. We define $\Gamma \upharpoonright X = \Gamma' \subseteq \Gamma$ where $\text{dom}(\Gamma') = \text{dom}(\Gamma) \cap X$.

Let \sqsubseteq be the reflexive transitive closure of the axioms $\tau_1 \cap \tau_2 \sqsubseteq \tau_1$ and $\tau_1 \cap \tau_2 \sqsubseteq \tau_2$. If $\Gamma = (x_i : \tau_i)_n$ and $\Gamma' = (x_i : \tau'_i)_n$ then $\Gamma \sqsubseteq \Gamma'$ iff for all $i \in \{1, \dots, n\}$, $\tau_i \sqsubseteq \tau'_i$.

| | | | |
|------------------------|---|--------------------|--|
| (ref) | $\tau \leq \tau$ | (Ω) | $\tau \leq \Omega$ |
| (tr) | $(\tau_1 \leq \tau_2 \wedge \tau_2 \leq \tau_3) \Rightarrow \tau_1 \leq \tau_3$ | $(\Omega' - lazy)$ | $\tau \rightarrow \Omega \leq \Omega \rightarrow \Omega$ |
| (in_L) | $\tau_1 \cap \tau_2 \leq \tau_1$ | $(idem)$ | $\tau \leq \tau \cap \tau$ |
| (in_R) | $\tau_1 \cap \tau_2 \leq \tau_2$ | $(\Omega - \eta)$ | $\Omega \leq \Omega \rightarrow \Omega$ |
| $(\rightarrow - \cap)$ | $(\tau_1 \rightarrow \tau_2) \cap (\tau_1 \rightarrow \tau_3) \leq \tau_1 \rightarrow (\tau_2 \cap \tau_3)$ | $(\Omega - lazy)$ | $\tau_1 \rightarrow \tau_2 \leq \Omega \rightarrow \Omega$ |
| (mon') | $(\tau_1 \leq \tau_2 \wedge \tau_1 \leq \tau_3) \Rightarrow \tau_1 \leq \tau_2 \cap \tau_3$ | | |
| (mon) | $(\tau_1 \leq \tau'_1 \wedge \tau_2 \leq \tau'_2) \Rightarrow \tau_1 \cap \tau_2 \leq \tau'_1 \cap \tau'_2$ | | |
| $(\rightarrow - \eta)$ | $(\tau_1 \leq \tau'_1 \wedge \tau'_2 \leq \tau_2) \Rightarrow \tau'_1 \rightarrow \tau'_2 \leq \tau_1 \rightarrow \tau_2$ | | |

Figure 1. The ordering axioms on types

3.
 - – Let $\nabla_1 = \{(ref), (tr), (in_L), (in_R), (\rightarrow - \cap), (mon'), (mon), (\rightarrow - \eta)\}$.
 - Let $\nabla_2 = \nabla_1 \cup \{(\Omega), (\Omega' - lazy)\}$.
 - Let $\nabla_D = \{(in_L), (in_R)\}$.
 - Let $\nabla_{D_I} = \nabla_D \cup \{(idem)\}$
 - – $\text{Type}^{\nabla_1} = \text{Type}^{\nabla_2} = \text{Type}^{\nabla_{D_I}} = \text{Type}^1$.
 - $\text{Type}^{\nabla_2} = \text{Type}^2$.
 - – Let ∇ be a set of axioms from Figure 1. The relation \leq^∇ is defined on types Type^∇ and axioms ∇ . We use \leq^1 instead of \leq^{∇_1} and \leq^2 instead of \leq^{∇_2} .
 - The equivalence relation is defined by: $\tau_1 \sim^\nabla \tau_2 \iff \tau_1 \leq^\nabla \tau_2 \wedge \tau_2 \leq^\nabla \tau_1$. We use \sim^1 instead of \sim^{∇_1} and \sim^2 instead of \sim^{∇_2} .
 - – Let $\lambda^{\cap 1}$ be the type system built on Λ , Type^1 and \vdash^1 such that \vdash^1 is the type derivability relation on \mathcal{B}^1 , Λ and Type^1 generated using the following typing rules of Figure 2: (ax) , (\rightarrow_E) , (\rightarrow_I) , (\cap_I) and (\leq^1) .
 - Let $\lambda^{\cap 2}$ be the type system built on Λ , Type^2 and \vdash^2 such that \vdash^2 is the type derivability relation on \mathcal{B}^2 , Λ and Type^2 generated using the following typing rules of Figure 2: (ax) , (\rightarrow_E) , (\rightarrow_I) , (\cap_I) , (\leq^2) and (Ω) .
 - Let \mathcal{D} be the type system built on Λ , Type^1 and $\vdash^{\beta\eta}$ where $\vdash^{\beta\eta}$ is the type derivability relation on \mathcal{B}^1 , Λ and Type^1 generated using the following typing rules of Figure 2: (ax) , (\rightarrow_E) , (\rightarrow_I) , (\cap_I) , (\cap_{E1}) and (\cap_{E2}) .
 - Let \mathcal{D}_I be the type system built on Λ , Type^1 and $\vdash^{\beta I}$ where $\vdash^{\beta I}$ is the type derivability relation on \mathcal{B}^1 , Λ and Type^1 generated using the following typing rule of Figure 2: (ax^I) , (\rightarrow_{E^I}) , (\rightarrow_I) , (\cap_I) , (\cap_{E1}) and (\cap_{E2}) . Moreover, in this type system, we assume that $\sigma \cap \sigma = \sigma$.

3. Problems of Ghilezan and Likavec's reducibility method [GL02]

In this section we introduce the reducibility method of [GL02] and show where exactly it fails. Throughout, we let $\otimes = \lambda x.xx$.

| | |
|--|---|
| $\frac{}{\Gamma, x : \tau \vdash x : \tau} (ax)$ | $\frac{}{x : \tau \vdash x : \tau} (ax^I)$ |
| $\frac{\Gamma \vdash M : \tau_1 \rightarrow \tau_2 \quad \Gamma \vdash N : \tau_1}{\Gamma \vdash MN : \tau_2} (\rightarrow_E)$ | $\frac{\Gamma_1 \vdash M : \tau_1 \rightarrow \tau_2 \quad \Gamma_2 \vdash N : \tau_1}{\Gamma_1 \cap \Gamma_2 \vdash MN : \tau_2} (\rightarrow_{EI})$ |
| $\frac{\Gamma, x : \tau_1 \vdash M : \tau_2}{\Gamma \vdash \lambda x. M : \tau_1 \rightarrow \tau_2} (\rightarrow_I)$ | $\frac{\Gamma \vdash M : \tau_1 \quad \Gamma \vdash M : \tau_2}{\Gamma \vdash M : \tau_1 \cap \tau_2} (\cap_I)$ |
| $\frac{\Gamma \vdash M : \tau_1 \cap \tau_2}{\Gamma \vdash M : \tau_1} (\cap_{E1})$ | $\frac{\Gamma \vdash M : \tau_1 \cap \tau_2}{\Gamma \vdash M : \tau_2} (\cap_{E2})$ |
| $\frac{\Gamma \vdash M : \tau_1 \quad \tau_1 \leq^\nabla \tau_2}{\Gamma \vdash M : \tau_2} (\leq^\nabla)$ | $\frac{}{\Gamma \vdash M : \Omega} (\Omega)$ |

Figure 2. The typing rules

Definition 3.1. (Type systems and reducibility of [GL02])

Let $i \in \{1, 2\}$. Let \mathcal{P} range over 2^Λ .

1. The type interpretation $\llbracket - \rrbracket_{\mathcal{P}}^i \in \mathbf{Type}^i \rightarrow 2^\Lambda \rightarrow 2^\Lambda$ is defined by:

- $\llbracket \alpha \rrbracket_{\mathcal{P}}^i = \mathcal{P}$.
- $\llbracket \tau_1 \cap \tau_2 \rrbracket_{\mathcal{P}}^i = \llbracket \tau_1 \rrbracket_{\mathcal{P}}^i \cap \llbracket \tau_2 \rrbracket_{\mathcal{P}}^i$.
- $\llbracket \Omega \rrbracket_{\mathcal{P}}^2 = \Lambda$.
- $\llbracket \sigma_1 \rightarrow \sigma_2 \rrbracket_{\mathcal{P}}^1 = \{M \mid \forall N \in \llbracket \sigma_1 \rrbracket_{\mathcal{P}}^1. MN \in \llbracket \sigma_2 \rrbracket_{\mathcal{P}}^1\}$.
- $\llbracket \tau_1 \rightarrow \tau_2 \rrbracket_{\mathcal{P}}^2 = \{M \in \mathcal{P} \mid \forall N \in \llbracket \tau_1 \rrbracket_{\mathcal{P}}^2, MN \in \llbracket \tau_2 \rrbracket_{\mathcal{P}}^2\}$.

2. A valuation of term variables in Λ is a function $\nu \in \mathcal{V} \rightarrow \Lambda$. We write $v(x := M)$ for the function v' where $v'(x) = M$ and $v'(y) = v(y)$ if $y \neq x$.

3. let ν be a valuation of term variables in Λ . Then $\llbracket - \rrbracket_{\nu} \in \Lambda \rightarrow \Lambda$ is defined by:
 $\llbracket M \rrbracket_{\nu} = M[x_1 := \nu(x_1), \dots, x_n := \nu(x_n)]$, where $FV(M) = \{x_1, \dots, x_n\}$.

- 4.
- $\nu \models_{\mathcal{P}}^i M : \tau$ iff $\llbracket M \rrbracket_{\nu} \in \llbracket \tau \rrbracket_{\mathcal{P}}^i$
 - $\nu \models_{\mathcal{P}}^i \Gamma$ iff $\forall (x : \tau) \in \Gamma. \nu(x) \in \llbracket \tau \rrbracket_{\mathcal{P}}^i$
 - $\Gamma \models_{\mathcal{P}}^i M : \tau$ iff $\forall \nu \in \mathcal{V} \rightarrow \Lambda. \nu \models_{\mathcal{P}}^i \Gamma \Rightarrow \nu \models_{\mathcal{P}}^i M : \tau$

5. Let $\mathcal{X} \subseteq \Lambda$. Let us recall the variable, saturation, closure and invariance under abstraction predicates defined by Ghilezan and Likavec:

- $\mathbf{VAR}^i(\mathcal{P}, \mathcal{X}) \iff \mathcal{V} \subseteq \mathcal{X}$.
- $\mathbf{SAT}^1(\mathcal{P}, \mathcal{X}) \iff$
 $(\forall M \in \Lambda. \forall x \in \mathcal{V}. \forall N \in \mathcal{P}. M[x := N] \in \mathcal{X} \Rightarrow (\lambda x. M)N \in \mathcal{X})$.

- $\text{SAT}^2(\mathcal{P}, \mathcal{X}) \iff (\forall M, N \in \Lambda. \forall x \in \mathcal{V}. M[x := N] \in \mathcal{X} \Rightarrow (\lambda x.M)N \in \mathcal{X}).$
- $\text{CLO}^1(\mathcal{P}, \mathcal{X}) \iff (\forall M \in \Lambda. \forall x \in \mathcal{V}. Mx \in \mathcal{X} \Rightarrow M \in \mathcal{P}).$
- $\text{CLO}^2(\mathcal{P}, \mathcal{X}) \iff \text{CLO}(\mathcal{P}, \mathcal{X}) \iff (\forall M \in \Lambda. \forall x \in \mathcal{V}. M \in \mathcal{X} \Rightarrow \lambda x.M \in \mathcal{P}).$
- $\text{VAR}(\mathcal{P}, \mathcal{X}) \iff (\forall x \in \mathcal{V}. \forall n \in \mathbb{N}. \forall N_1, \dots, N_n \in \mathcal{P}. xN_1 \dots N_n \in \mathcal{X}).$
- $\text{SAT}(\mathcal{P}, \mathcal{X}) \iff (\forall M, N \in \Lambda. \forall x \in \mathcal{V}. \forall n \in \mathbb{N}. \forall N_1, \dots, N_n \in \mathcal{P}. M[x := N]N_1 \dots N_n \in \mathcal{X} \Rightarrow (\lambda x.M)NN_1 \dots N_n \in \mathcal{X}).$
- $\text{INV}(\mathcal{P}) \iff (\forall M \in \Lambda. \forall x \in \mathcal{V}. M \in \mathcal{P} \iff \lambda x.M \in \mathcal{P}).$

For $\mathcal{R} \in \{\text{VAR}^i, \text{SAT}^i, \text{CLO}^i\}$, let $\mathcal{R}(\mathcal{P}) \iff \forall \tau \in \text{Type}^i. \mathcal{R}(\mathcal{P}, \llbracket \tau \rrbracket_{\mathcal{P}}^i)$.

Lemma 3.1. (Basic lemmas proved in [GL02])

1. (a) $\llbracket M \rrbracket_{\nu(x:=N)} \equiv \llbracket M \rrbracket_{\nu(x:=x)}[x := N]$
 (b) $\llbracket MN \rrbracket_{\nu} \equiv \llbracket M \rrbracket_{\nu} \llbracket N \rrbracket_{\nu}$
 (c) $\llbracket \lambda x.M \rrbracket_{\nu} \equiv \lambda x. \llbracket M \rrbracket_{\nu(x:=x)}$
2. If $\text{VAR}^1(\mathcal{P})$ and $\text{CLO}^1(\mathcal{P})$ then
 - (a) for all $\sigma \in \text{Type}^1$, $\llbracket \sigma \rrbracket_{\mathcal{P}}^1 \subseteq \mathcal{P}$.
 - (b) if $\text{SAT}^1(\mathcal{P})$ and $\Gamma \vdash^1 M : \sigma$ then $\Gamma \Vdash_{\mathcal{P}}^1 M : \sigma$ and $M \in \mathcal{P}$
3. For all $\tau \in \text{Type}^2$, if $\tau \not\leq^2 \Omega$ then $\llbracket \tau \rrbracket_{\mathcal{P}}^2 \subseteq \mathcal{P}$
4. If $\tau_1 \leq^2 \tau_2$ then $\llbracket \tau_1 \rrbracket_{\mathcal{P}}^2 \subseteq \llbracket \tau_2 \rrbracket_{\mathcal{P}}^2$.
5. If $\text{VAR}^2(\mathcal{P})$, $\text{SAT}^2(\mathcal{P})$ and $\text{CLO}^2(\mathcal{P})$ then $\Gamma \vdash^2 M : \tau$ implies $\Gamma \Vdash_{\mathcal{P}}^2 M : \tau$
6. If $\text{VAR}^2(\mathcal{P})$, $\text{SAT}^2(\mathcal{P})$ and $\text{CLO}^2(\mathcal{P})$ then for all $\tau \in \text{Type}^2$, if $\tau \not\leq^2 \Omega$ and $\Gamma \vdash^2 M : \tau$ then $M \in \mathcal{P}$
7. $\text{CLO}(\mathcal{P}, \mathcal{P}) \Rightarrow \forall \tau \in \text{Type}^2. \tau \not\leq^2 \Omega \Rightarrow \text{CLO}^2(\mathcal{P}, \llbracket \tau \rrbracket_{\mathcal{P}}^2)$.

Proof:

We only prove 5. By induction on $\Gamma \vdash^2 M : \tau$. (ax) and (Ω) are easy. (\cap_I) (resp. (\rightarrow_E) resp. (\leq^2)) is by IH (resp. IH and 1, resp. IH and 4).

(\rightarrow_I) By IH, $\Gamma, x : \tau_1 \Vdash_{\mathcal{P}}^2 M : \tau_2$. Let $\nu \Vdash_{\mathcal{P}}^2 \Gamma$ and $N \in \llbracket \tau_1 \rrbracket_{\mathcal{P}}^2$. Then $\nu(x := N) \Vdash_{\mathcal{P}}^2 \Gamma$ since $x \notin \text{dom}(\Gamma)$ and $\nu(x := N) \Vdash_{\mathcal{P}}^2 x : \tau_1$ since $N \in \llbracket \tau_1 \rrbracket_{\mathcal{P}}^2$. Therefore $\nu(x := N) \Vdash_{\mathcal{P}}^2 M : \tau_2$, i.e. $\llbracket M \rrbracket_{\nu(x:=N)} \in \llbracket \tau_2 \rrbracket_{\mathcal{P}}^2$. Hence, by lemma 3.1.1, $\llbracket M \rrbracket_{\nu(x:=x)}[x := N] \in \llbracket \tau_2 \rrbracket_{\mathcal{P}}^2$. By $\text{SAT}^2(\mathcal{P})$, we get $(\lambda x. \llbracket M \rrbracket_{\nu(x:=x)})N \in \llbracket \tau_2 \rrbracket_{\mathcal{P}}^2$. Again by lemma 3.1.1, $(\llbracket \lambda x.M \rrbracket_{\nu})N \in \llbracket \tau_2 \rrbracket_{\mathcal{P}}^2$. Hence $\llbracket \lambda x.M \rrbracket_{\nu} \in \{M \mid \forall N \in \llbracket \tau_1 \rrbracket_{\mathcal{P}}^2. MN \in \llbracket \tau_2 \rrbracket_{\mathcal{P}}^2\}$.

By $\text{VAR}^2(\mathcal{P})$, $x \in \llbracket \tau_1 \rrbracket_{\mathcal{P}}^2$, hence by the same argument as above we obtain $\llbracket M \rrbracket_{\nu(x:=x)} \in \llbracket \tau_2 \rrbracket_{\mathcal{P}}^2$. So by $\text{CLO}^2(\mathcal{P})$, $\lambda x. \llbracket M \rrbracket_{\nu(x:=x)} \in \mathcal{P}$ and by lemma 3.1.1, $\llbracket \lambda x.M \rrbracket_{\nu} \in \mathcal{P}$. Hence, we conclude that $\llbracket \lambda x.M \rrbracket_{\nu} \in \llbracket \tau_1 \rightarrow \tau_2 \rrbracket_{\mathcal{P}}^2$.

□

Ghilezan and Likavec claim that if $\text{CLO}^1(\mathcal{P})$, $\text{VAR}^1(\mathcal{P})$ and $\text{SAT}^1(\mathcal{P})$ are true then $\text{SN}_\beta \subseteq \mathcal{P}$ (note that this result does not make any use of the type system $\lambda\Omega^1$).

After giving the above definitions and lemmas, [GL02] states that since the predicates $(\text{VAR}^i(\mathcal{P}))$, $\text{SAT}^i(\mathcal{P})$ and $\text{CLO}^i(\mathcal{P})$ for $i \in \{1, 2\}$ have been shown to be sufficient to develop the reducibility method, and since in order to prove these predicates one needs stronger induction hypotheses which are easier to prove, the paper sets out to show that these stronger conditions when $i = 2$ are the three predicates $\text{VAR}(\mathcal{P}, \mathcal{P})$, $\text{SAT}(\mathcal{P}, \mathcal{P})$ and $\text{CLO}(\mathcal{P}, \mathcal{P})$. However, as we show below, this attempt fails. They do not develop the necessary stronger induction hypotheses for the case when $i = 1$, and $\lambda\Omega^1$ can only anyway type strongly normalisable terms, so we will not consider the case $i = 1$ further.

Definition 3.2 and lemma 3.2 are necessary to establish the results of this section (the failure of the method of [GL02]). In definition 3.2, we use the fact that the defined preorder relation is commutative, associative and idempotent:

Remark 3.1. Commutativity, associativity and idempotence w.r.t. the preorder relation are given by the axioms (in_L) , (in_R) , (mon') , (tr) and (ref) listed in figure 1.

Definition 3.2. Let $to \in \text{TypeOmega} ::= \Omega \mid to_1 \cap to_2$.

Let $\text{inInter}(\tau, \tau')$ be true iff $\tau = \tau'$ or $\tau' = \tau_1 \cap \tau_2$ and $(\text{inInter}(\tau, \tau_1)$ or $\text{inInter}(\tau, \tau_2))$.

By commutativity and associativity we write $\tau_1 \cap \dots \cap \tau_n$, where $n \geq 1$, for any type τ such that $(\text{inInter}(\tau_0, \tau))$ iff there exists $i \in \{1, \dots, n\}$ such that $\tau_0 = \tau_i$.

Lemma 3.2. 1. If $\tau_1 \leq^\Omega \tau_2$ and $\tau_1 \in \text{TypeOmega}$ then $\tau_2 \in \text{TypeOmega}$.

2. If $\tau \leq^\Omega \tau'$ and $\tau' \not\leq^2 \Omega$ then $\tau \not\leq^2 \Omega$.

3. If $\tau \cap \tau' \not\leq^2 \Omega$ then $\tau \not\leq^2 \Omega$ or $\tau' \not\leq^2 \Omega$.

4. If $\tau' \sim^2 \Omega$ then $\tau \leq^\Omega \tau \cap \tau'$

5. If $\tau \leq^\Omega \tau'$ and $\text{inInter}(\tau_1 \rightarrow \tau_2, \tau')$ and $\tau_2 \not\leq^2 \Omega$ then there exist $n \geq 1$ and $\tau'_1, \tau''_1, \dots, \tau'_n, \tau''_n$ such that for all $i \in \{1, \dots, n\}$, $\text{inInter}(\tau'_i \rightarrow \tau''_i, \tau)$ and $\tau''_i \not\leq^2 \Omega$ and $\tau'_1 \cap \dots \cap \tau''_n \leq^\Omega \tau_2$. Moreover, if $\tau_1 \sim^2 \Omega$ then for all $i \in \{1, \dots, n\}$, $\tau'_i \sim^2 \Omega$.

6. For all $\tau, \tau' \in \text{Type}^2$, $\alpha \rightarrow \Omega \rightarrow \tau' \not\leq^2 \Omega \rightarrow \tau$

Proof:

1. By induction on the size derivation of $\tau_1 \leq^\Omega \tau_2$ and then by case on the last rule of the derivation.

2. Let $\tau \leq^\Omega \tau'$. Assume $\tau \sim^2 \Omega$. Then $\Omega \leq^\Omega \tau$ and by transitivity $\Omega \leq^\Omega \tau'$. Moreover, by (Ω) , $\tau' \leq^\Omega \Omega$. So $\tau' \sim^2 \Omega$.

3. By (Ω) , $\tau \cap \tau' \leq^\Omega \Omega$. let $\tau \sim^2 \Omega$ and $\tau' \sim^2 \Omega$, so $\Omega \leq^\Omega \tau$ and $\Omega \leq^\Omega \tau'$ and by (mon') , $\Omega \leq^\Omega \tau \cap \tau'$.

4. By (Ω) , $\tau \leq^\Omega \Omega$ and by transitivity, $\tau \leq^\Omega \tau'$ because $\Omega \leq^\Omega \tau'$. By (ref) , $\tau \leq^\Omega \tau$ and by (mon') , $\tau \leq^\Omega \tau \cap \tau'$.
5. By induction on the size derivation of $\tau \leq^\Omega \tau'$ and then by case on the last rule of the derivation.
6. let $\tau' \in \mathbf{Type}^2$. First we prove that $\Omega \rightarrow \tau' \not\leq^2 \Omega$. Assume $\Omega \rightarrow \tau' \leq^2 \Omega$ then $\Omega \leq^\Omega \Omega \rightarrow \tau'$. By lemma 3.2.1, $\Omega \rightarrow \tau' \in \mathbf{TypeOmega}$ which is false.
 Let $\tau \sim^2 \Omega$. Assume $\alpha \rightarrow \Omega \rightarrow \tau' \sim^2 \Omega \rightarrow \tau$ then $\Omega \rightarrow \tau \leq^\Omega \alpha \rightarrow \Omega \rightarrow \tau'$. By lemma 3.2.5, $\tau \leq^\Omega \Omega \rightarrow \tau'$ which is false.
 Let $\tau \not\sim^2 \Omega$. Assume $\alpha \rightarrow \Omega \rightarrow \tau' \sim^2 \Omega \rightarrow \tau$ then $\alpha \rightarrow \Omega \rightarrow \tau' \leq^\Omega \Omega \rightarrow \tau$. By lemma 3.2.5, $\alpha \sim^2 \Omega$ because $\Omega \sim^2 \Omega$, which is false.

□

The next lemma establishes the failure of a basic lemma of [GL02].

Lemma 3.3. (Lemma 3.16 of [GL02] is false)

Lemma 3.16 of [GL02] stated below is false:

$$\mathbf{VAR}(\mathcal{P}, \mathcal{P}) \Rightarrow \forall \tau \in \mathbf{Type}^2. (\forall \tau' \in \mathbf{Type}^2. (\tau \not\leq^2 \Omega \rightarrow \tau') \Rightarrow \mathbf{VAR}(\mathcal{P}, \llbracket \tau \rrbracket_{\mathcal{P}}^2)).$$

Proof:

To show that the above statement is false, we give the following counterexample. Note that $\mathbf{VAR}(\mathcal{P}, \llbracket \tau \rrbracket_{\mathcal{P}}^2) \Rightarrow \mathcal{V} \subseteq \llbracket \tau \rrbracket_{\mathcal{P}}^2$. Let $x \in \mathcal{V}$, τ be $\alpha \rightarrow \Omega \rightarrow \alpha$ and \mathcal{P} be \mathbf{WN}_β . By lemma 3.2.6, for all $\tau' \in \mathbf{Type}^2$, $\tau \not\leq^2 \Omega \rightarrow \tau'$ and $\mathbf{VAR}(\mathcal{P}, \mathcal{P})$ is true. Assume $\mathbf{VAR}(\mathcal{P}, \llbracket \tau \rrbracket_{\mathcal{P}}^2)$, then $x \in \llbracket \tau \rrbracket_{\mathcal{P}}^2$. Then $x \in \llbracket \alpha \rightarrow \Omega \rightarrow \alpha \rrbracket_{\mathcal{P}}^2 = \llbracket \tau \rrbracket_{\mathcal{P}}^2$ because $x \in \mathcal{P} = \llbracket \alpha \rrbracket_{\mathcal{P}}^2$, and $xx(\otimes \otimes) \in \llbracket \alpha \rrbracket_{\mathcal{P}}^2 = \mathcal{P}$ because $\otimes \otimes \in \Lambda = \llbracket \Omega \rrbracket_{\mathcal{P}}^2$. But $xx(\otimes \otimes) \in \mathcal{P}$ is false, so $\mathbf{VAR}(\mathcal{P}, \llbracket \tau \rrbracket_{\mathcal{P}}^2)$ is false. □

The proof for Lemma 3.18 of [GL02] does not work (because of a wrong use of an induction hypothesis) but we have not yet proved or disproved that lemma:

Remark 3.2. (It is not clear that Lemma 3.18 of [GL02] holds)

It is not clear whether this lemma of [GL02] holds: $\mathbf{SAT}(\mathcal{P}, \mathcal{P}) \Rightarrow \forall \tau \in \mathbf{Type}^2. (\forall \tau' \in \mathbf{Type}^2. (\tau \not\leq^2 \Omega \rightarrow \tau') \Rightarrow \mathbf{SAT}(\mathcal{P}, \llbracket \tau \rrbracket_{\mathcal{P}}^2))$.

The proof given in [GL02] does not go through and we have neither been able to prove nor disprove this lemma. It remains that this lemma is not yet proved and hence cannot be used in further proofs.

Then, Ghilezan and Likavec give a proposition (Proposition 3.21) which is the reducibility method for typable terms. However, the proof of that proposition depends on two problematic lemmas (lemma 3.16 which we showed to fail in our lemma 3.3, and lemma 3.18 which according to remark 3.2 has not been proved). First, here is a lemma:

Lemma 3.4. $\mathbf{VAR}(\mathbf{WN}_\beta, \mathbf{WN}_\beta)$, $\mathbf{CLO}(\mathbf{WN}_\beta, \mathbf{WN}_\beta)$, $\mathbf{INV}(\mathbf{WN}_\beta)$ and $\mathbf{SAT}(\mathbf{WN}_\beta, \mathbf{WN}_\beta)$ hold.

Proof:

- $\mathbf{VAR}(\mathbf{WN}_\beta, \mathbf{WN}_\beta)$ holds because $\forall x \in \mathcal{V}, \forall n \geq 0, \forall N_1, \dots, N_n \in \mathbf{WN}_\beta, xN_1 \dots N_n \in \mathbf{WN}_\beta$.

- $\text{CLO}(\text{WN}_\beta, \text{WN}_\beta)$ holds, because if $\exists n, m \geq 0, \exists x_0 \in \mathcal{V}, \exists N_1, \dots, N_m \in \text{NF}_\beta$ such that $M \rightarrow_\beta^* \lambda x_1 \dots \lambda x_n. x_0 N_1 \dots N_m$ then $\forall y \in \mathcal{V}, \lambda y. M \rightarrow_\beta^* \lambda y. \lambda x_1 \dots \lambda x_n. x_0 N_1 \dots N_m \in \text{NF}_\beta$.
 $\text{INV}(\text{WN}_\beta)$ holds, because if $\exists n, m \geq 0, \exists x_0 \in \mathcal{V}, \exists N_1, \dots, N_m \in \text{NF}_\beta$ such that $\lambda x. M \rightarrow_\beta^* \lambda x_1 \dots \lambda x_n. x_0 N_1 \dots N_m$ then $x_1 = y$ and $M \rightarrow_\beta^* \lambda x_2 \dots \lambda x_n. x_0 N_1 \dots N_m$
- $\text{SAT}(\text{WN}_\beta, \text{WN}_\beta)$ holds, since if $M[x := N]N_1 \dots N_n \in \text{WN}_\beta$ where $n \geq 0$ and $N_1, \dots, N_n \in \text{WN}_\beta$ then $\exists P \in \text{NF}_\beta$ such that $M[x := N]N_1 \dots N_n \rightarrow_\beta^* P$.
Hence, $(\lambda x. M)N_1 \dots N_n \rightarrow_\beta M[x := N]N_1 \dots N_n \rightarrow_\beta^* P$. □

Lemma 3.5. (Proposition 3.21 of [GL02] fails)

Assume $\text{VAR}(\mathcal{P}, \mathcal{P})$, $\text{SAT}(\mathcal{P}, \mathcal{P})$ and $\text{CLO}(\mathcal{P}, \mathcal{P})$. It is **not** the case that: $\forall \tau \in \text{Type}^2. (\tau \not\sim^2 \Omega \wedge \forall \tau' \in \text{Type}^2. (\tau \not\sim^2 \Omega \rightarrow \tau') \wedge \Gamma \vdash^2 M : \tau \Rightarrow M \in \mathcal{P})$.

Proof:

Let \mathcal{P} be WN_β . Note that $\lambda y. \lambda z. \otimes \otimes \notin \text{WN}_\beta$ and $\emptyset \vdash^2 \lambda y. \lambda z. \otimes \otimes : \alpha \rightarrow \Omega \rightarrow \Omega$ is derivable, where $\alpha \rightarrow \Omega \rightarrow \Omega \not\sim^2 \Omega$ and by lemma 3.2.6, $\alpha \rightarrow \Omega \rightarrow \Omega \not\sim^2 \Omega \rightarrow \tau'$, for all $\tau' \in \text{Type}^2$. Since $\text{VAR}(\text{WN}_\beta, \text{WN}_\beta)$, $\text{CLO}(\text{WN}_\beta, \text{WN}_\beta)$ and $\text{SAT}(\text{WN}_\beta, \text{WN}_\beta)$ hold, we get a counterexample for Proposition 3.21 of [GL02]. □

Finally, also Ghilezan and Likavec's proof method for untyped terms fails.

Lemma 3.6. (Proposition 3.23 of [GL02] fails)

Proposition 3.23 of [GL02] which states that "If $\mathcal{P} \subseteq \Lambda$ is invariant under abstraction (i.e., $\text{INV}(\mathcal{P})$), $\text{VAR}(\mathcal{P}, \mathcal{P})$ and $\text{SAT}(\mathcal{P}, \mathcal{P})$ then $\mathcal{P} = \Lambda$ " fails.

Proof:

The proof given in [GL02] depends on Proposition 3.21 which fails.

As $\text{VAR}(\text{WN}_\beta, \text{WN}_\beta)$, $\text{SAT}(\text{WN}_\beta, \text{WN}_\beta)$ and $\text{INV}(\text{WN}_\beta)$, we get a counterexample for Proposition 3.23. □

4. How much of the reducibility method of [GL02] can we salvage?

In this section, we give some indications on the limits of the method. We show how these limits affect the salvation of the method, we partially salvage it and we show that this can now be correctly used to establish confluence, standardisation and weak head normal forms but only for restricted sets of lambda terms and types (that we believe to be equal to the set of strongly normalisable terms). We also point out some links between the work done by Ghilezan and Likavec and the work done by Gallier.

Because we proved that Proposition 3.23 of [GL02] is false, we know that the set of properties that a set of terms \mathcal{P} has to fulfil in order to be equal to the set of terms of the untyped λ -calculus cannot be $\text{INV}(\mathcal{P})$, $\text{VAR}(\mathcal{P}, \mathcal{P})$ and $\text{SAT}(\mathcal{P}, \mathcal{P})$. So even if one works on the soundness result or on the type interpretation (the set of realisers), to obtain the same result as the one claimed by Ghilezan and Likavec, one should come up with a new set of properties.

Proposition 3.23 of [GL02] states a set of properties characterising the set of terms of the untyped λ -calculus. The predicate $\text{VAR}(\Lambda, \Lambda)$ states that the variables (and the terms of the form $xNM_1 \cdots M_n$) belong to the untyped λ -calculus. The predicate $\text{INV}(\Lambda)$ states among other things that if a term is a λ -term then the abstraction of a variable over this term is a λ -term too. To get a full characterisation of the set of terms of the untyped λ -calculus, a predicate, say $\text{APP}(\mathcal{P})$, stating that $(\lambda x.M)NM_1 \cdots M_n \in \mathcal{P}$ if $M, N, M_1, \dots, M_n \in \mathcal{P}$, needs to hold. Note that this predicate cannot be equivalent to the sum of properties $\text{VAR}(\mathcal{P}, \mathcal{P})$, $\text{SAT}(\mathcal{P}, \mathcal{P})$ and $\text{INV}(\mathcal{P})$ since we saw that the set WN_β satisfies these properties but is not equal to the λ -calculus. Hence, these properties are not enough to characterise the λ -calculus.

The problem with these properties is that if one tries to salvage Ghilezan and Likavec's reducibility method, the properties $\text{VAR}(\mathcal{P}, \mathcal{P})$ and $\text{CLO}(\mathcal{P}, \mathcal{P})$ will impose a restriction on the arrow types for which the interpretation is in \mathcal{P} (the realisers of arrow types). This can be seen in the arrow type case of the proofs of lemmas 4.1.5 and 4.2. We show at the end of this section that even if the obtained result when considering these restrictions is different from (in some sens, is an improvement of) the one given by Ghilezan and Likavec using the type system $\lambda\cap^1$, we do not succeed in salvaging their method.

The use of the non-trivial types (we recall the definition below) introduced by Gallier [Gal03] are not much help in this case, because of the precise restriction imposed by $\text{VAR}(\mathcal{P}, \mathcal{P})$. One might also want to consider the sets of properties (we do not recall them in this paper for lack of space) stated in his work [Gal03], but which are unfortunately not easy to prove for CR , because a proof of $xM \in \text{CR}$ for all $M \in \Lambda$ is required. Moreover, if one succeeds in proving that the variables are included in the interpretation of a defined set of types containing $\Omega \rightarrow \alpha$, where Ω is interpreted as Λ and α as \mathcal{P} , then one has proved that $xM \in \mathcal{P}$, so that in the case $\mathcal{P} = \text{CR}$, we have $M \in \text{CR}$.

It is worth pointing out that a part of the work done by Gallier [Gal03] would still be valid if adapted to the type system $\lambda\cap^2$. Gallier defines the non-trivial types as follows:

$$\psi \in \text{NonTrivial} ::= \alpha \mid \tau \rightarrow \psi \mid \tau \cap \psi \mid \psi \cap \tau$$

Types in Type^2 are then interpreted as follows: $\llbracket \alpha \rrbracket_{\mathcal{P}} = \mathcal{P}$, $\llbracket \psi \cap \tau \rrbracket_{\mathcal{P}} = \llbracket \tau \cap \psi \rrbracket_{\mathcal{P}} = \llbracket \tau \rrbracket_{\mathcal{P}} \cap \llbracket \psi \rrbracket_{\mathcal{P}}$, $\llbracket \tau \rrbracket_{\mathcal{P}} = \Lambda$ if $\tau \notin \text{NonTrivial}$ and $\llbracket \tau \rightarrow \psi \rrbracket_{\mathcal{P}} = \{M \in \mathcal{P} \mid \forall N \in \llbracket \tau \rrbracket_{\mathcal{P}}. MN \in \llbracket \psi \rrbracket_{\mathcal{P}}\}$. We can easily prove that if $\tau_1 \leq^2 \tau_2$ then $\llbracket \tau_1 \rrbracket_{\mathcal{P}} \subseteq \llbracket \tau_2 \rrbracket_{\mathcal{P}}$. Hence, considering the type system $\lambda\cap^2$ instead of the type system $\mathcal{D}\Omega$, the method of Gallier gets a set of predicates which when satisfied by a set of terms \mathcal{P} implies that the set of terms typable in the system $\lambda\cap^2$ by a non-trivial type is a subset of \mathcal{P} . Gallier proved that the set of head-normalising λ -terms satisfies each of the given predicates.

Using a method similar to Ghilezan and Likavec's method, Gallier proved also that the set of weakly head-normalising terms (W) is equal to the set of terms typable by a weakly non-trivial type in the type system $\mathcal{D}\Omega$. The set of weakly non-trivial types is defined as follows:

$$\psi \in \text{WeaklyNonTrivial} ::= \alpha \mid \tau \rightarrow \psi \mid \Omega \rightarrow \Omega \mid \tau \cap \psi \mid \psi \cap \tau$$

As explained above, we can try and salvage Ghilezan and Likavec's method by first restricting the set of realisers when defining the interpretation of the set of types in Type^2 . The different restrictions lead us to the definition of Type^3 and the following type interpretation:

Definition 4.1. $\rho \in \text{Type}^3 ::= \alpha \mid \tau \rightarrow \rho \mid \rho \cap \tau \mid \tau \cap \rho$.

- $\llbracket \alpha \rrbracket_{\mathcal{P}}^3 = \mathcal{P}$.

- $\llbracket \tau_1 \cap \tau_2 \rrbracket_{\mathcal{P}}^3 = \llbracket \tau_1 \rrbracket_{\mathcal{P}}^3 \cap \llbracket \tau_2 \rrbracket_{\mathcal{P}}^3$, if $\tau_1 \cap \tau_2 \in \mathbf{Type}^3$.
- $\llbracket \tau \rrbracket_{\mathcal{P}}^3 = \Lambda$, if $\tau \notin \mathbf{Type}^3$.
- $\llbracket \tau_1 \rightarrow \tau_2 \rrbracket_{\mathcal{P}}^3 = \{M \in \mathcal{P} \mid \forall N \in \llbracket \tau_1 \rrbracket_{\mathcal{P}}^3. MN \in \llbracket \tau_2 \rrbracket_{\mathcal{P}}^3\}$, if $\tau_1 \rightarrow \tau_2 \in \mathbf{Type}^3$.

In order to prove the relation between the stronger induction hypotheses (VAR, SAT and CLO, and particularly the variable one) and the ones depending on type interpretations (VAR², SAT² and CLO²), and in order to be able to use these stronger induction hypotheses in the soundness lemma, we have to impose other restrictions.

Definition 4.2. We let $\varphi \in \mathbf{Type}^4 ::= \alpha \mid \Omega \mid \rho \rightarrow \varphi \mid \varphi \cap \tau \mid \tau \cap \varphi$.

We let $\Gamma \in \mathcal{B}^3 = \{\{x_1 : \varphi_1, \dots, x_n : \varphi_n\} \mid \forall i, j \in \{1, \dots, n\}. x_i = x_j \Rightarrow \varphi_i = \varphi_j\}$

Let \vdash^3 be the relation \vdash^2 where (ax) is replaced by (ax') and \mathcal{B}^2 is replaced by \mathcal{B}^3 . Let $\lambda \cap^3$ be the type system $\lambda \cap^2$ where (ax) is replaced by (ax') and \mathcal{B}^2 is replaced by \mathcal{B}^3 . Let $\models_{\mathcal{P}}^3$ be the relation $\models_{\mathcal{P}}^2$ where $\llbracket \tau \rrbracket_{\mathcal{P}}^2$ is replaced by $\llbracket \tau \rrbracket_{\mathcal{P}}^3$.

Due to the saturation predicate and its uses, we could impose some other restrictions on the type system. Another alternative is to slightly modify this predicate (in order not to burden ourselves with another notation for the saturation predicate, we keep the same name):

Definition 4.3. $\text{SAT}(\mathcal{P}, \mathcal{X}) \iff (\forall M, N \in \Lambda. \forall x \in \mathcal{V}. \forall n \in \mathbb{N}. \forall N_1, \dots, N_n \in \Lambda. M[x := N]N_1 \dots N_n \in \mathcal{X} \Rightarrow (\lambda x.M)NN_1 \dots N_n \in \mathcal{X})$.

We can prove that if $\mathcal{P} \in \{\mathbf{CR}, \mathbf{S}, \mathbf{W}\}$ then $\text{SAT}(\mathcal{P}, \mathcal{P})$ holds.

The next lemma is useful to prove soundness. Particularly useful is the relation between the old and the new induction hypothesis.

- Lemma 4.1.**
1. $\llbracket \tau_1 \cap \tau_2 \rrbracket_{\mathcal{P}}^3 = \llbracket \tau_1 \rrbracket_{\mathcal{P}}^3 \cap \llbracket \tau_2 \rrbracket_{\mathcal{P}}^3$.
 2. $\llbracket \rho \rrbracket_{\mathcal{P}}^3 \subseteq \mathcal{P}$.
 3. If $\tau_1 \leq^3 \tau_2$ and $\tau_2 \in \mathbf{Type}^3$ then $\tau_1 \in \mathbf{Type}^3$.
 4. If $\tau_1 \leq^2 \tau_2$ then $\llbracket \tau_1 \rrbracket_{\mathcal{P}}^3 \subseteq \llbracket \tau_2 \rrbracket_{\mathcal{P}}^3$.
 5. If $\text{VAR}(\mathcal{P}, \mathcal{P})$ then for all $\varphi \in \mathbf{Type}^4$, $\text{VAR}(\mathcal{P}, \llbracket \varphi \rrbracket_{\mathcal{P}}^3)$.
 6. If $\text{SAT}(\mathcal{P}, \mathcal{P})$ then for all $\tau \in \mathbf{Type}^2$, $\text{SAT}(\mathcal{P}, \llbracket \tau \rrbracket_{\mathcal{P}}^3)$.

Proof:

1. If $\tau_1 \cap \tau_2 \in \mathbf{Type}^3$ then it is done by definition. Otherwise $\tau_1, \tau_2 \notin \mathbf{Type}^3$. Hence $\llbracket \tau_1 \cap \tau_2 \rrbracket_{\mathcal{P}}^3 = \Lambda = \Lambda \cap \Lambda = \llbracket \tau_1 \rrbracket_{\mathcal{P}}^3 \cap \llbracket \tau_2 \rrbracket_{\mathcal{P}}^3$.
2. By induction on the structure of τ .
3. By induction on the size of the derivation of $\tau_1 \leq^2 \tau_2$ and then by case on the last step.
4. By induction on the size of the derivation of $\tau_1 \leq^2 \tau_2$ and then by case on the last step.

5. Assume $\text{VAR}(\mathcal{P}, \mathcal{P})$. Let $n \geq 0$, $x \in \mathcal{V}$ and for all $i \in \{1, \dots, n\}$, $M_i \in \mathcal{P}$. By the hypothesis, $xM_1 \cdots M_n \in \mathcal{P}$. We prove that $xM_1 \cdots M_n \in \llbracket \varphi \rrbracket_{\mathcal{P}}^3$ by induction on the structure of φ .
6. Assume $\text{SAT}(\mathcal{P}, \mathcal{P})$. Let $n \geq 0$, $x \in \mathcal{V}$, $M, N \in \Lambda$ and for all $i \in \{1, \dots, n\}$, $N_i \in \Lambda$. We prove that if $M[x := N]N_1 \cdots N_n \in \llbracket \tau \rrbracket_{\mathcal{P}}^3$ then $(\lambda x.M)NN_1 \cdots N_n \in \llbracket \tau \rrbracket_{\mathcal{P}}^3$ by induction on the structure of τ .

□

We now state the soundness lemma:

Lemma 4.2. If $\text{VAR}(\mathcal{P}, \mathcal{P})$, $\text{SAT}(\mathcal{P}, \mathcal{P})$, $\text{CLO}(\mathcal{P}, \mathcal{P})$ and $\Gamma \vdash^3 M : \tau$ then $\Gamma \models_{\mathcal{P}}^3 M : \tau$

Proof:

By induction on the size of the derivation of $\Gamma \vdash^3 M : \tau$ and then by case on the last rule used in the derivation. Cases dealing with $\tau \notin \text{Type}^3$ are trivial since $\llbracket \tau \rrbracket_{\mathcal{P}}^3 = \Lambda$. So we only consider $\tau \in \text{Type}^3$.

- (ax) : Let $\nu \models_{\mathcal{P}}^3 \Gamma, x : \varphi$ then $\nu(x) \in \llbracket \varphi \rrbracket_{\mathcal{P}}^3$.
- (\rightarrow_E) : By IH, $\Gamma \models_{\mathcal{P}}^3 M : \tau_1 \rightarrow \tau_2$ and $\Gamma \models_{\mathcal{P}}^2 N : \tau_1$, so by lemma 3.1.1, $\Gamma \models_{\mathcal{P}}^3 MN : \tau_2$ (because if $\tau_2 \in \text{Type}^3$ then $\tau_1 \rightarrow \tau_2 \in \text{Type}^3$).
- (\rightarrow_I) : By IH, $\Gamma, x : \tau_1 \models_{\mathcal{P}}^3 M : \tau_2$. Let $\nu \models_{\mathcal{P}}^3 \Gamma$ and $N \in \llbracket \tau_1 \rrbracket_{\mathcal{P}}^3$. Then $\nu(x := N) \models_{\mathcal{P}}^3 \Gamma$ since $x \notin \text{dom}(\Gamma)$ and $\nu(x := N) \models_{\mathcal{P}}^3 x : \tau_1$ since $N \in \llbracket \tau_1 \rrbracket_{\mathcal{P}}^3$. Therefore $\nu(x := N) \models_{\mathcal{P}}^3 M : \tau_2$, i.e. $\llbracket M \rrbracket_{\nu(x:=N)} \in \llbracket \tau_2 \rrbracket_{\mathcal{P}}^3$. Hence, by lemma 3.1.1, $\llbracket M \rrbracket_{\nu(x:=x)}[x := N] \in \llbracket \tau_2 \rrbracket_{\mathcal{P}}^3$. Hence by applying $\text{SAT}(\mathcal{P}, \mathcal{P})$ and 4.1.6, we get $(\lambda x. \llbracket M \rrbracket_{\nu(x:=x)})N \in \llbracket \tau_2 \rrbracket_{\mathcal{P}}^3$. Again by lemma 3.1.1, $(\llbracket \lambda x.M \rrbracket_{\nu})N \in \llbracket \tau_2 \rrbracket_{\mathcal{P}}^3$. Hence $\llbracket \lambda x.M \rrbracket_{\nu} \in \{M \mid \forall N \in \llbracket \tau_1 \rrbracket_{\mathcal{P}}^3. MN \in \llbracket \tau_2 \rrbracket_{\mathcal{P}}^3\}$.
Since $\tau_1 \in \text{Type}^4$, by $\text{VAR}(\mathcal{P}, \mathcal{P})$ and 4.1.5, $x \in \llbracket \tau_1 \rrbracket_{\mathcal{P}}^3$, hence by the same argument as above we obtain $\llbracket M \rrbracket_{\nu(x:=x)} \in \llbracket \tau_2 \rrbracket_{\mathcal{P}}^3$. Since $\tau_1 \rightarrow \tau_2 \in \text{Type}^3$ then $\tau_2 \in \text{Type}^3$, so by $\text{CLO}(\mathcal{P}, \mathcal{P})$ and 4.1.2, $\lambda x. \llbracket M \rrbracket_{\nu(x:=x)} \in \mathcal{P}$ and by lemma 3.1.1, $\llbracket \lambda x.M \rrbracket_{\nu} \in \mathcal{P}$. Hence, we conclude that $\llbracket \lambda x.M \rrbracket_{\nu} \in \llbracket \tau_1 \rightarrow \tau_2 \rrbracket_{\mathcal{P}}^3$.
- (\leq^3) : We conclude by IH and 4.1.4
- (Ω) : This case is trivial because $\Omega \notin \text{Type}^3$.

□

The next lemma states that the set of terms satisfying the Church-Rosser, the weak head normalisation or the standardisation properties satisfies the variable, saturation and closure predicates.

Lemma 4.3. Let $\mathcal{P} \in \{\text{CR}, \text{S}, \text{W}\}$. Then $\text{VAR}(\mathcal{P}, \mathcal{P})$, $\text{SAT}(\mathcal{P}, \mathcal{P})$ and $\text{CLO}(\mathcal{P}, \mathcal{P})$.

We obtain the following proof method.

Proposition 4.1. If $\Gamma \vdash^3 M : \rho$ then $M \in \text{CR}$, $M \in \text{S}$, and $M \in \text{W}$.

Proof:

By lemma 4.3, lemma 4.1.2 and lemma 4.2

□

However, we strongly believe that the set of terms typable in our type system with a type ρ is no more than the set of strongly normalisable terms.

5. Reducibility method for the CR proofs w.r.t. βI - and $\beta\eta$ -reductions

In this section, we introduce the reducibility semantics for both βI - and $\beta\eta$ -reduction and establish its soundness (lemma 5.3). Then, we show that all typable terms satisfy the Church-Rosser property and that all terms of ΛI_c (resp. $\Lambda\eta_c$) are typable in system \mathcal{D}_I (resp. \mathcal{D}).

We start by introducing a reducibility semantics for types:

Definition 5.1. 1. Let $r \in \{\beta I, \beta\eta\}$. We define the type interpretation $\llbracket - \rrbracket^r : \mathbf{Type}^1 \rightarrow 2^\Lambda$ by:

- $\llbracket \alpha \rrbracket^r = \mathbf{CR}^r$, where $\alpha \in \mathcal{A}$.
- $\llbracket \sigma \cap \tau \rrbracket^r = \llbracket \sigma \rrbracket^r \cap \llbracket \tau \rrbracket^r$.
- $\llbracket \sigma \rightarrow \tau \rrbracket^r = \{M \in \mathbf{CR}^r \mid \forall N \in \llbracket \sigma \rrbracket^r. MN \in \llbracket \tau \rrbracket^r\}$.

2. A set $\mathcal{X} \subseteq \Lambda$ is saturated iff $\forall n \geq 0. \forall M, N, M_1, \dots, M_n \in \Lambda. \forall x \in \mathcal{V}$.

$$M[x := N]M_1 \dots M_n \in \mathcal{X} \Rightarrow (\lambda x.M)NM_1 \dots M_n \in \mathcal{X}$$

3. A set $\mathcal{X} \subseteq \Lambda I$ is I-saturated iff $\forall n \geq 0. \forall M, N, M_1, \dots, M_n \in \Lambda. \forall x \in \mathcal{V}$.

$$x \in \text{fv}(M) \Rightarrow M[x := N]M_1 \dots M_n \in \mathcal{X} \Rightarrow (\lambda x.M)NM_1 \dots M_n \in \mathcal{X}$$

Here is a background lemma which is familiar to many type systems:

Lemma 5.1. 1. If $\Gamma \vdash^{\beta I} M : \sigma$ then $M \in \Lambda I$ and $\text{fv}(M) = \text{dom}(\Gamma)$.

2. Let $\Gamma \vdash^{\beta\eta} M : \sigma$. Then $\text{fv}(M) \subseteq \text{dom}(\Gamma)$ and if $\Gamma \subseteq \Gamma'$ then $\Gamma' \vdash^{\beta\eta} M : \sigma$.

3. Let $r \in \{\beta I, \beta\eta\}$. If $\Gamma \vdash^r M : \sigma$, $\sigma \sqsubseteq \sigma'$ and $\Gamma' \sqsubseteq \Gamma$ then $\Gamma' \vdash^r M : \sigma'$.

The next lemma states that the interpretations of types are saturated and only contain terms that are Church-Rosser. Krivine [Kri90] proved a similar result for $r = \beta$ and where \mathbf{CR}_0^r and \mathbf{CR}^r were replaced by the corresponding sets of strongly normalising terms. Koletsos and Stavrinou [KS08] adapted Krivine's lemma for Church-Rosser w.r.t. β -reduction instead of strong normalisation. Here, we adapt the result to βI and $\beta\eta$.

Lemma 5.2. Let $r \in \{\beta I, \beta\eta\}$.

1. $\forall \sigma \in \mathbf{Type}^1. \mathbf{CR}_0^r \subseteq \llbracket \sigma \rrbracket^r \subseteq \mathbf{CR}^r$.
2. $\mathbf{CR}^{\beta I}$ is I-saturated.
3. $\mathbf{CR}^{\beta\eta}$ is saturated.
4. $\forall \sigma \in \mathbf{Type}^1. \llbracket \sigma \rrbracket^{\beta I}$ is I-saturated.
5. $\forall \sigma \in \mathbf{Type}^1. \llbracket \sigma \rrbracket^{\beta\eta}$ is saturated.

Next we adapt the soundness lemma of [Kri90] to both $\vdash^{\beta I}$ and $\vdash^{\beta\eta}$.

Lemma 5.3. Let $r \in \{\beta I, \beta\eta\}$. If $x_1 : \sigma_1, \dots, x_n : \sigma_n \vdash^r M : \sigma$ and $\forall i \in \{1, \dots, n\}, N_i \in \llbracket \sigma_i \rrbracket^r$ then $M[(x_i := N_i)_1^n] \in \llbracket \sigma \rrbracket^r$.

Finally, we adapt a corollary from [KS08] to show that every term of Λ typable in system \mathcal{D}_I (resp. \mathcal{D}) has the βI (resp. $\beta\eta$) Church-Rosser property.

Corollary 5.1. Let $r \in \{\beta I, \beta\eta\}$. If $\Gamma \vdash^r M : \sigma$ then $M \in \mathbf{CR}^r$.

Proof:

Let $\Gamma = (x_i : \sigma_i)_n$. By lemma 5.2, $\forall i \in \{1, \dots, n\}, x_i \in \llbracket \sigma_i \rrbracket^r$, so by lemma 5.3 and again by lemma 5.2, $M \in \llbracket \sigma \rrbracket^r \subseteq \mathbf{CR}^r$. \square

To accommodate βI - and $\beta\eta$ -reduction, the next lemma generalises a lemma given in [Kri90] (and used in [KS08]). This lemma states that every term of ΛI_c (resp. $\Lambda\eta_c$) is typable in system \mathcal{D}_I (resp. \mathcal{D}).

Lemma 5.4. Let $\text{fv}(M) \setminus \{c\} = \{x_1, \dots, x_n\} \subseteq \text{dom}(\Gamma)$ where $c \notin \text{dom}(\Gamma)$.

1. If $M \in \Lambda I_c$ then for $\Gamma' = \Gamma \upharpoonright \text{fv}(M)$, $\exists \sigma, \tau \in \mathbf{Type}^1$ such that if $c \in \text{fv}(M)$ then $\Gamma', c : \sigma \vdash^{\beta I} M : \tau$, and if $c \notin \text{fv}(M)$ then $\Gamma' \vdash^{\beta I} M : \tau$.
2. If $M \in \Lambda\eta_c$ then $\exists \sigma, \tau \in \mathbf{Type}^1$ such that $\Gamma, c : \sigma \vdash^{\beta\eta} M : \tau$.

6. Adapting the CR proof of Koletsos and Stavrinou [KS08] to βI -reduction

Koletsos and Stavrinou [KS08] gave a proof of Church-Rosser for β -reduction for the intersection type system \mathcal{D} of Definition 2.4 (studied in detail by Krivine in [Kri90]) and showed that this can be used to establish confluence of β -developments without using strong normalisation. In this section, we adapt their proof to βI . First, we adapt and formalise a number of definitions and lemmas given by Krivine in [Kri90] in order to make them applicable to βI -developments. Then, we adapt [KS08] to establish the confluence of βI -developments and hence of βI -reduction.

6.1. Formalising βI -developments

The next definition, taken from [Kri90] (and used in [KS08]) uses the variable c to destroy the βI -redexes of M which are not in the set \mathcal{F} of βI -redex occurrences in M , and to neutralise applications so that they cannot be transformed into redexes after βI -reduction. For example, in $c(\lambda x.x)y$, c is used to destroy the βI -redex $(\lambda x.x)y$.

Definition 6.1. ($\Phi^c(-, -)$)

Let $M \in \Lambda I$, such that $c \notin \text{fv}(M)$ and $\mathcal{F} \subseteq \mathcal{R}_M^{\beta I}$.

1. If $M = x$ then $\mathcal{F} = \emptyset$ and $\Phi^c(x, \mathcal{F}) = x$
2. If $M = \lambda x.N$ such that $x \neq c$ and $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}_N^{\beta I}$ then $\Phi^c(\lambda x.N, \mathcal{F}) = \lambda x.\Phi^c(N, \mathcal{F}')$
3. If $M = NP$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}_N^{\beta I}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq \mathcal{R}_P^{\beta I}$ then

$$\Phi^c(NP, \mathcal{F}) = \begin{cases} c\Phi^c(N, \mathcal{F}_1)\Phi^c(P, \mathcal{F}_2) & \text{if } 0 \notin \mathcal{F} \\ \Phi^c(N, \mathcal{F}_1)\Phi^c(P, \mathcal{F}_2) & \text{otherwise} \end{cases}$$

The next lemma is an adapted version of a lemma which appears in [KS08] and which in turns adapts a lemma from [Kri90].

Lemma 6.1. 1. If $M \in \Lambda I$, $c \notin \text{fv}(M)$, and $\mathcal{F} \subseteq \mathcal{R}_M^{\beta I}$ then

- (a) $\text{fv}(M) = \text{fv}(\Phi^c(M, \mathcal{F})) \setminus \{c\}$.
- (b) $\Phi^c(M, \mathcal{F}) \in \Lambda I_c$.
- (c) $|\Phi^c(M, \mathcal{F})|^c = M$.
- (d) $|\langle \Phi^c(M, \mathcal{F}), \mathcal{R}_{\Phi^c(M, \mathcal{F})}^{\beta I} \rangle|^c = \mathcal{F}$.

2. Let $M \in \Lambda I_c$.

- (a) $|\langle M, \mathcal{R}_M^{\beta I} \rangle|^c \subseteq \mathcal{R}_{|M|^c}^{\beta I}$ and $M = \Phi^c(|M|^c, |\langle M, \mathcal{R}_M^{\beta I} \rangle|^c)$.
- (b) $\langle |M|^c, |\langle M, \mathcal{R}_M^{\beta I} \rangle|^c \rangle$ is the one and only pair $\langle N, \mathcal{F} \rangle$ such that $N \in \Lambda I$, $c \notin \text{fv}(N)$, $\mathcal{F} \subseteq \mathcal{R}_N^{\beta I}$ and $\Phi^c(N, \mathcal{F}) = M$.

The next lemma is needed to define βI -developments.

Lemma 6.2. Let $M \in \Lambda I$, such that $c \notin \text{fv}(M)$, $\mathcal{F} \subseteq \mathcal{R}_M^{\beta I}$, $p \in \mathcal{F}$ and $M \xrightarrow{p}_{\beta I} M'$. Then, there exists a unique set $\mathcal{F}' \subseteq \mathcal{R}_{M'}^{\beta I}$, such that $\Phi^c(M, \mathcal{F}) \xrightarrow{p'}_{\beta I} \Phi^c(M', \mathcal{F}')$ and $|\langle \Phi^c(M, \mathcal{F}), p' \rangle|^c = p$.

We follow [Kri90] and define the set of βI -residuals of a set of βI -redexes \mathcal{F} relative to a sequence of βI -redexes. First, we give the definition relative to one redex.

Definition 6.2. Let $M \in \Lambda I$, such that $c \notin \text{fv}(M)$, $\mathcal{F} \subseteq \mathcal{R}_M^{\beta I}$, $p \in \mathcal{F}$ and $M \xrightarrow{p}_{\beta I} M'$. By lemma 6.2, there exists a unique $\mathcal{F}' \subseteq \mathcal{R}_{M'}^{\beta I}$ such that $\Phi^c(M, \mathcal{F}) \xrightarrow{p'}_{\beta I} \Phi^c(M', \mathcal{F}')$ and $|\langle \Phi^c(M, \mathcal{F}), p' \rangle|^c = p$. We call \mathcal{F}' the set of βI -residuals in M' of the set of βI -redexes \mathcal{F} in M relative to p .

Definition 6.3. (βI -development)

Let $M \in \Lambda I$ where $c \notin \text{fv}(M)$ and $\mathcal{F} \subseteq \mathcal{R}_M^{\beta I}$. A one-step βI -development of $\langle M, \mathcal{F} \rangle$, denoted $\langle M, \mathcal{F} \rangle \rightarrow_{\beta Id} \langle M', \mathcal{F}' \rangle$, is a βI -reduction $M \xrightarrow{p}_{\beta I} M'$ where $p \in \mathcal{F}$ and \mathcal{F}' is the set of βI -residuals in M' of the set of βI -redexes \mathcal{F} in M relative to p . A βI -development is the transitive closure of a one-step βI -development. We write also $M \xrightarrow{\mathcal{F}}_{\beta Id} M_n$ for the βI -development $\langle M, \mathcal{F} \rangle \rightarrow_{\beta Id}^* \langle M_n, \mathcal{F}_n \rangle$.

6.2. Confluence of βI -developments, hence of βI -reduction

The next lemma is informative about developments.

Lemma 6.3. 1. Let $M \in \Lambda I$, such that $c \notin \text{fv}(M)$ and $\mathcal{F} \subseteq \mathcal{R}_M^{\beta I}$. Then: $\langle M, \mathcal{F} \rangle \rightarrow_{\beta Id}^* \langle M', \mathcal{F}' \rangle \iff \Phi^c(M, \mathcal{F}) \rightarrow_{\beta Id}^* \Phi^c(M', \mathcal{F}')$.

2. Let $M \in \Lambda I$, such that $c \notin \text{fv}(M)$ and $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{R}_M^{\beta I}$. If $\langle M, \mathcal{F}_1 \rangle \rightarrow_{\beta Id} \langle M', \mathcal{F}'_1 \rangle$ then there exists $\mathcal{F}'_2 \subseteq \mathcal{R}_{M'}^{\beta I}$ such that $\mathcal{F}'_1 \subseteq \mathcal{F}'_2$ and $\langle M, \mathcal{F}_2 \rangle \rightarrow_{\beta Id} \langle M', \mathcal{F}'_2 \rangle$.

The next lemma adapts the main theorem in [KS08] where as far as we know it first appeared.

Lemma 6.4. (confluence of the βI -developments)

Let $M \in \Lambda I$, such that $c \notin \text{fv}(M)$. If $M \xrightarrow{\mathcal{F}_1}_{\beta Id} M_1$ and $M \xrightarrow{\mathcal{F}_2}_{\beta Id} M_2$, then there exist $\mathcal{F}'_1 \subseteq \mathcal{R}_{M_1}^{\beta I}$, $\mathcal{F}'_2 \subseteq \mathcal{R}_{M_2}^{\beta I}$ and $M_3 \in \Lambda I$ such that $M_1 \xrightarrow{\mathcal{F}'_1}_{\beta Id} M_3$ and $M_2 \xrightarrow{\mathcal{F}'_2}_{\beta Id} M_3$.

We follow [Bar84] and [KS08] and define one reduction as follows:

Notation 6.1. Let $M, M' \in \Lambda I$, such that $c \notin \text{fv}(M)$. We define one reduction by: $M \rightarrow_{1I} M' \iff \exists \mathcal{F}, \mathcal{F}', (M, \mathcal{F}) \rightarrow_{\beta Id}^* (M', \mathcal{F}')$.

Before establishing the main result of this section we need the following lemma:

Lemma 6.5. 1. Let $c \notin \text{fv}(M)$. Then, $\mathcal{R}_{\Phi^c(M, \emptyset)}^{\beta I} = \emptyset$.

2. Let $c \notin \text{fv}(MN)$ and $x \neq c$. Then, $\mathcal{R}_{\Phi^c(M, \emptyset)[x := \Phi^c(N, \emptyset)]}^{\beta I} = \emptyset$.

3. Let $c \notin \text{fv}(M)$. If $p \in \mathcal{R}_M^{\beta I}$ and $\Phi^c(M, \{p\}) \rightarrow_{\beta I} M'$ then $\mathcal{R}_{M'}^{\beta I} = \emptyset$.

4. Let $M \in \Lambda I$ such that $c \notin \text{fv}(M)$. If $M \xrightarrow{p}_{\beta I} M'$ then $\langle M, \{p\} \rangle \rightarrow_{\beta Id} \langle M', \emptyset \rangle$.

5. $\rightarrow_{\beta I}^* = \rightarrow_{1I}^*$.

Finally, we achieve what we started to do: the confluence of βI -reduction on ΛI .

Lemma 6.6. $\Lambda I \subseteq \text{CR}^{\beta I}$.

7. Generalisation of the method to $\beta\eta$ -reduction

In this section, we generalise the method of [KS08] to handle $\beta\eta$ -reduction. This generalisation is not trivial since we needed to define developments involving η -reduction and to establish the important result of the closure under η -reduction of a defined set of frozen terms. These were the main reasons that led us to extend the various definitions related to developments. For example, clause (R4) of the definition of $\Lambda\eta_c$ in Definition 2.2 aims to ensure closure under η -reduction. The definition of Λ_c in [Kri90] excluded such a rule and hence we lose closure under η -reduction as can be seen by the following example: Let $M = \lambda x.cNx \in \Lambda_c$ where $x \notin \text{fv}(N)$ and $N \in \Lambda_c$, then $M \rightarrow_{\eta} cN \notin \Lambda_c$.

First, we formalise $\beta\eta$ -residuals and $\beta\eta$ -developments in section 7.1. Then, we compare our notion of $\beta\eta$ -residuals with those of Curry and Feys [CF58] and Klop [Klo80] in section 7.2, establishing that we allow less residuals than Klop but we believe more residuals than Curry and Feys. Finally, we establish in section 7.3 the confluence of $\beta\eta$ -developments and hence of $\beta\eta$ -reduction.

7.1. Formalising $\beta\eta$ -developments

The next two definitions adapt definition 6.1 to deal with $\beta\eta$ -reduction. The variable c enables to destroy the $\beta\eta$ -redexes of M which are not in the set \mathcal{F} of $\beta\eta$ -redex occurrences in M ; to neutralise applications so that they cannot be transformed into redexes after $\beta\eta$ -reduction; and to neutralise bound variables so λ -abstraction cannot be transformed into redexes after $\beta\eta$ -reduction. For example, in $\lambda x.y(c(cx))$ ($x \neq c$), c is used to destroy the η -redex $\lambda x.yx$.

Definition 7.1. $(\Psi^c(-, -), \Psi_0^c(-, -))$

Let $c \notin \text{fv}(M)$ and $\mathcal{F} \subseteq \mathcal{R}_M^{\beta\eta}$.

(P1) If $M \in \mathcal{V} \setminus \{c\}$ then $\mathcal{F} \stackrel{2,3}{=} \emptyset$ and

$$\Psi^c(M, \mathcal{F}) = \{c^n(M) \mid n > 0\} \quad \Psi_0^c(M, \mathcal{F}) = \{M\}$$

(P2) If $M = \lambda x.N$ and $x \neq c$ and $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}_N^{\beta\eta}$:

$$\Psi^c(M, \mathcal{F}) = \begin{cases} \{c^n(\lambda x.P[x := c(cx)]) \mid n \geq 0 \wedge P \in \Psi^c(N, \mathcal{F}')\} & \text{if } 0 \notin \mathcal{F} \\ \{c^n(\lambda x.N') \mid n \geq 0 \wedge N' \in \Psi_0^c(N, \mathcal{F}')\} & \text{otherwise} \end{cases}$$

$$\Psi_0^c(M, \mathcal{F}) = \begin{cases} \{\lambda x.N'[x := c(cx)] \mid N' \in \Psi^c(N, \mathcal{F}')\} & \text{if } 0 \notin \mathcal{F} \\ \{\lambda x.N' \mid N' \in \Psi_0^c(N, \mathcal{F}')\} & \text{otherwise} \end{cases}$$

(P3) If $M = NP$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}_N^{\beta\eta}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq \mathcal{R}_P^{\beta\eta}$ then:

$$\Psi^c(M, \mathcal{F}) = \begin{cases} \{c^n(cN'P') \mid n \geq 0 \wedge N' \in \Psi^c(N, \mathcal{F}_1) \wedge P' \in \Psi^c(P, \mathcal{F}_2)\} & \text{if } 0 \notin \mathcal{F} \\ \{c^n(N'P') \mid n \geq 0 \wedge N' \in \Psi_0^c(N, \mathcal{F}_1) \wedge P' \in \Psi^c(P, \mathcal{F}_2)\} & \text{otherwise} \end{cases}$$

$$\Psi_0^c(M, \mathcal{F}) = \begin{cases} \{cN'P' \mid N' \in \Psi^c(N, \mathcal{F}_1) \wedge P' \in \Psi_0^c(P, \mathcal{F}_2)\} & \text{if } 0 \notin \mathcal{F} \\ \{N'P' \mid N' \in \Psi_0^c(N, \mathcal{F}_1) \wedge P' \in \Psi_0^c(P, \mathcal{F}_2)\} & \text{otherwise} \end{cases}$$

The next lemma is needed to define $\beta\eta$ -developments.

Lemma 7.1. 1. Let $c \notin \text{fv}(M)$ and $\mathcal{F} \subseteq \mathcal{R}_M^{\beta\eta}$. We have:

- (a) $\Psi_0^c(M, \mathcal{F}) \subseteq \Psi^c(M, \mathcal{F})$.
- (b) $\forall N \in \Psi^c(M, \mathcal{F}). \text{fv}(M) = \text{fv}(N) \setminus \{c\}$.
- (c) $\Psi^c(M, \mathcal{F}) \subseteq \Lambda\eta_c$.
- (d) Let $M = Nx$ such that $x \notin \text{fv}(N) \cup \{c\}$ and $P \in \Psi_0^c(M, \mathcal{F})$. Then, $\mathcal{R}_{\lambda x.P}^{\beta\eta} = \{0\} \cup \{1.p \mid p \in \mathcal{R}_P^{\beta\eta}\}$.
- (e) Let $M = Nx$. If $Px \in \Psi^c(Nx, \mathcal{F})$ then $Px \in \Psi_0^c(Nx, \mathcal{F})$.
- (f) $\forall N \in \Psi^c(M, \mathcal{F}). \forall n \geq 0. c^n(N) \in \Psi^c(M, \mathcal{F})$.
- (g) $\forall N \in \Psi^c(M, \mathcal{F}). |N|^c = M$.
- (h) $\forall N \in \Psi^c(M, \mathcal{F}). \mathcal{F} = |\langle N, \mathcal{R}_N^{\beta\eta} \rangle|^c$.

2. Let $M \in \Lambda\eta_c$. We have:

- (a) $|\langle M, \mathcal{R}_M^{\beta\eta} \rangle|^c \subseteq \mathcal{R}_{|M|^c}^{\beta\eta}$ and $M \in \Psi^c(|M|^c, |\langle M, \mathcal{R}_M^{\beta\eta} \rangle|^c)$.
- (b) $\langle |M|^c, |\langle M, \mathcal{R}_M^{\beta\eta} \rangle|^c \rangle$ is the one and only pair $\langle N, \mathcal{F} \rangle$ such that $c \notin \text{fv}(N)$, $\mathcal{F} \subseteq \mathcal{R}_N^{\beta\eta}$ and $M \in \Psi^c(N, \mathcal{F})$.

3. Let $M \in \Lambda$, such that $c \notin \text{fv}(M)$, $\mathcal{F} \subseteq \mathcal{R}_M^{\beta\eta}$, $p \in \mathcal{F}$ and $M \xrightarrow{p}_{\beta\eta} M'$. Then, there exists a unique set $\mathcal{F}' \subseteq \mathcal{R}_{M'}^{\beta\eta}$, such that for all $N \in \Psi^c(M, \mathcal{F})$ there exists $N' \in \Psi^c(M', \mathcal{F}')$ and $p' \in \mathcal{R}_N^{\beta\eta}$ such that $N \xrightarrow{p'}_{\beta\eta} N'$ and $|\langle N, p' \rangle|^c = p$.

Definition 7.2. ($\beta\eta$ -development)

1. Let $M \in \Lambda$, $\mathcal{F} \subseteq \mathcal{R}_M^{\beta\eta}$, $p \in \mathcal{F}$ and $M \xrightarrow{p}_{\beta\eta} M'$. By lemma 7.1.3, there exists a unique $\mathcal{F}' \subseteq \mathcal{R}_{M'}^{\beta\eta}$, such that for all $N \in \Psi^c(M, \mathcal{F})$, there exist $N' \in \Psi^c(M', \mathcal{F}')$ and $p' \in \mathcal{R}_N^{\beta\eta}$ such that $N \xrightarrow{p'}_{\beta\eta} N'$ and $|\langle N, p' \rangle|^c = p$. We call \mathcal{F}' the set of $\beta\eta$ -residuals in M' of the set of $\beta\eta$ -redexes \mathcal{F} in M relative to p .
2. Let $M \in \Lambda$, where $c \notin \text{fv}(M)$, and $\mathcal{F} \subseteq \mathcal{R}_M^{\beta\eta}$. A one-step $\beta\eta$ -development of $\langle M, \mathcal{F} \rangle$, denoted $\langle M, \mathcal{F} \rangle \rightarrow_{\beta\eta d} \langle M', \mathcal{F}' \rangle$, is a $\beta\eta$ -reduction $M \xrightarrow{p}_{\beta\eta} M'$ where $p \in \mathcal{F}$ and \mathcal{F}' is the set of $\beta\eta$ -residuals in M' of the set of $\beta\eta$ -redexes \mathcal{F} in M relative to p . A $\beta\eta$ -development is the transitive closure of a one-step $\beta\eta$ -development. We write also $M \xrightarrow{\mathcal{F}}_{\beta\eta d} M'$ for the $\beta\eta$ -development $\langle M, \mathcal{F} \rangle \rightarrow_{\beta\eta d}^* \langle M', \mathcal{F}' \rangle$.

7.2. Comparison with Curry and Feys [CF58] and Klop [Klo80]

A full common definition of a $\beta\eta$ -residual is given by Curry and Feys [CF58] (p. 117, 118). Another definition of $\beta\eta$ -residual (called λ -residual) is presented by Klop [Klo80] (definition 2.4, p. 254). Klop [Klo80] shows that each definition enables to prove different properties of developments. Following the definition of a $\beta\eta$ -residual given by Curry and Feys [CF58] (and as pointed out in [CF58, Klo80, BBKV76]), if the η -redex $\lambda x.(\lambda y.M)x$, where $x \notin \text{fv}(\lambda y.M)$, is reduced in the term $P = (\lambda x.(\lambda y.M)x)N$ to give the term $Q = (\lambda y.M)N$, then Q is not a $\beta\eta$ -residual of P in P (note that following the definition of a λ -residual given by Klop [Klo80], Q is a λ -residual of the redex $(\lambda y.M)x$ in P since the λ of the redex Q is the same as the λ of the redex $(\lambda y.M)x$ in P). Moreover, if the β -redex $(\lambda y.My)x$, where $y \notin \text{fv}(M)$, is reduced in the term $P = \lambda x.(\lambda y.My)x$ to give the term $Q = \lambda x.Mx$, then Q is not a $\beta\eta$ -residual of P in P (note that following the definition of a λ -residual given by Klop [Klo80], Q is a λ -residual of the redex P in P since the λ of the redex Q is the same as the λ of the redex P in P). Our definition 7.2.1 differs from the common one stated by Curry and Feys [CF58] by these cases as we illustrate in the following example: $\Psi^c((\lambda x.(\lambda y.M)x)N, \{1, 1.0, 1.1.0\}) = \{c^n((\lambda x.(\lambda y.P[y := c(cy)]))x)Q) \mid n \geq 0 \wedge P \in \Psi^c(M, \emptyset) \wedge Q \in \Psi^c(N, \emptyset)\}$, where $x \notin \text{fv}(\lambda y.M)$. Let $p = 1.0$ then $(\lambda x.(\lambda y.M)x)N \xrightarrow{p}_{\beta\eta} (\lambda y.M)N$.

Moreover, $P_0 = c^n((\lambda x.(\lambda y.P[y := c(cy)]))x)Q \xrightarrow{p'}_{\beta\eta} c^n((\lambda y.P[y := c(cy)]))Q$ such that $n \geq 0$, $P \in \Psi^c(M, \emptyset)$, $Q \in \Psi^c(N, \emptyset)$, $|\langle P_0, p' \rangle|^c = |\langle P_0, 2^n.1.0 \rangle|^c = p$ (using a lemma stated and proved in the long version of this article) and $c^n((\lambda y.P[y := c(cy)]))Q \in \Psi^c((\lambda y.M)N, \{0\})$.

Let us now compare our definition of $\beta\eta$ -residuals to the one given by Klop [Klo80] (λ -residuals). We believe that we accept more redexes as residuals of a set of redexes than Curry and Feys [CF58] (as shown by our examples given earlier on in section 7) and less than Klop.

We introduce the two calculi $\bar{\Lambda}$ and $\bar{\Lambda}_{\eta c}$ which are labelled versions of the calculi Λ and $\Lambda_{\eta c}$:

$$\begin{aligned}
t &\in \bar{\Lambda} & ::= & x \mid \lambda_n x.t \mid t_1 t_2 \\
v &\in \mathbf{ABS}_c & ::= & \lambda_n \bar{x}.w\bar{x} \mid \lambda_n \bar{x}.u[\bar{x} := c(c\bar{x})], \text{ where } \bar{x} \notin \text{fv}(w) \\
w &\in \mathbf{APP}_c & ::= & v \mid cu \\
u &\in \bar{\Lambda}_{\eta_c} & ::= & \bar{x} \mid v \mid wu \mid cu
\end{aligned}$$

where $\bar{x}, \bar{y} \in \mathcal{V} \setminus \{c\}$. Note that $\mathbf{ABS}_c \subseteq \mathbf{APP}_c \subseteq \bar{\Lambda}_{\eta_c} \subseteq \bar{\Lambda}$.

The labels enable to distinguish two different occurrences of a λ .

Since these two calculus are only labelled versions of Λ and Λ_{η_c} , let us assume in this section that the work done so far is true when Λ and Λ_{η_c} are replaced by $\bar{\Lambda}$ and $\bar{\Lambda}_{\eta_c}$.

Klop [Klo80] defines his λ -residuals as follows:

“Let $\mathcal{R} = M_0 \rightarrow M_1 \rightarrow \dots \rightarrow M_k \rightarrow \dots$ be a $\beta\eta$ -reduction, R_0 a redex in M_0 and R_k a redex in M_k such that the head- λ of R_k descends from that of R_0 .

Regardless whether R_0, R_k are β - or η -redexes, R_k is called a λ -residual of R_0 via \mathcal{R} .”

We define the head- λ of a $\beta\eta$ -redex by: $\text{headlam}((\lambda_n x.t_1)t_2) = \langle 1, n \rangle$ and $\text{headlam}(\lambda_n x.t_0 x) = \langle 2, n \rangle$, if $x \notin \text{fv}(t_0)$. If $\mathcal{F} \subseteq \mathcal{R}_t^{\beta\eta}$ we define $\text{headlamred}(t, \mathcal{F})$ to be $\{\langle i, n \rangle \mid \exists p \in \mathcal{F}. \text{headlam}(t|_p) = \langle i, n \rangle\}$. We define $\text{hlr}(t)$ to be $\text{headlamred}(t, \mathcal{R}_t^{\beta\eta})$.

The following lemma states the equality between the head- λ 's of a set \mathcal{F} of $\beta\eta$ -redexes of a term t and the head- λ 's of the $\beta\eta$ -redexes of any term u in the application of the function Ψ^c to t and \mathcal{F} :

Lemma 7.2. Let $c \notin \text{fv}(t)$ and $\mathcal{F} \subseteq \mathcal{R}_t^{\beta\eta}$. If $u \in \Psi^c(t, \mathcal{F})$ then $\text{hlr}(u) = \text{headlamred}(t, \mathcal{F})$.

The following lemma states that if a term u_1 in $\bar{\Lambda}_{\eta_c}$ reduces to a term u' then the set of head- λ 's of the $\beta\eta$ -redexes of u' is included in the set of head- λ 's of the $\beta\eta$ -redexes of u_1 .

Lemma 7.3. If $u_1 \in \bar{\Lambda}_{\eta_c}$ and $u_1 \xrightarrow{p}_{\beta\eta} u'$ then $\text{hlr}(u') \subseteq \text{hlr}(u_1)$.

Let us now prove that, following our definition, the set of head- λ 's of the $\beta\eta$ -residuals of a set of $\beta\eta$ -redexes in a term is included in the set of head- λ 's of the considered set of $\beta\eta$ -redexes.

Let $c \notin \text{fv}(t)$, $\mathcal{F} \subseteq \mathcal{R}_t^{\beta\eta}$ and $t \xrightarrow{p}_{\beta\eta} t'$ then by definition 7.2.1, there exists a unique $\mathcal{F}' \subseteq \mathcal{R}_{t'}^{\beta\eta}$, such that for all $u \in \Psi^c(t, \mathcal{F})$ (by lemma 7.1.1c, $u \in \bar{\Lambda}_{\eta_c}$), there exist $u' \in \Psi^c(t', \mathcal{F}')$ and $p' \in \mathcal{R}_u^{\beta\eta}$ such that $u \xrightarrow{p'}_{\beta\eta} u'$ and $|\langle u, p' \rangle|^c = p$. The set \mathcal{F}' is the set of $\beta\eta$ -residuals in t' of the set of redexes \mathcal{F} in t relative to p . By lemma 2.1.3, $c \notin \text{fv}(t')$. By definition $\Psi^c(t, \mathcal{F})$ is not empty. Let $u \in \Psi^c(t, \mathcal{F})$ then there exist $u' \in \Psi^c(t', \mathcal{F}')$ and $p' \in \mathcal{R}_u^{\beta\eta}$ such that $u \xrightarrow{p'}_{\beta\eta} u'$ and $|\langle u, p' \rangle|^c = p$. By lemma 7.3, $\text{hlr}(u') \subseteq \text{hlr}(u)$. So, by lemma 7.2, $\text{headlamred}(t', \mathcal{F}') \subseteq \text{headlamred}(t, \mathcal{F})$.

However, this is not enough to match Klop's definition of λ -residuals. As a matter of fact, we can find t and \mathcal{F} such that, following Klop's definition, $p_0 \in \mathcal{R}_{t'}^{\beta\eta}$ and p_0 is a λ -residual of \mathcal{F} via p but $p_0 \notin \mathcal{F}'$. For example: Let $t = (\lambda_0 x.xy)(\lambda_1 z.yz) \xrightarrow{0}_{\beta\eta} (\lambda_1 z.yz)y = t'$. Let $\mathcal{F} = \{0, 2.0\}$. Then $\Psi^c(t, \mathcal{F}) = \{c^{n_1}((\lambda_0 x.c^{n_2}(c^3(x)y))(c^{n_3}(\lambda_1 z.c^{n_4+1}(y)z))) \mid n_1, n_2, n_3, n_4 \geq 0\}$. Let $u \in \Psi^c(t, \mathcal{F})$, then $u = c^{n_1}((\lambda_0 x.c^{n_2}(c^3(x)y))(c^{n_3}(\lambda_1 z.c^{n_4+1}(y)z)))$ such that $n_1, n_2, n_3, n_4 \geq 0$. We obtain $u = c^{n_1}((\lambda_0 x.c^{n_2}(c^3(x)y))(c^{n_3}(\lambda_1 z.c^{n_4+1}(y)z))) \xrightarrow{p_0}_{\beta\eta} c^{n_1+n_2}(c^{n_3+3}(\lambda_1 z.c^{n_4+1}(y)z)y) = u'$ such that $p_0 = 2^{n_1}.0$. Then $\mathcal{F}' = \{1.0\}$ is the set of $\beta\eta$ -residuals in t' of the set of redexes \mathcal{F} in t relative to p . But 0 is a λ -residual of \mathcal{F} via 0 and $0 \notin \mathcal{F}'$.

It turns out that, though our $\beta\eta$ -residuals are λ -residuals, the opposite does not hold. For example: $t = \lambda_n \bar{x}.(\lambda_m \bar{y}.z\bar{y})\bar{x} \xrightarrow{1.0}_\beta \lambda_n \bar{x}.z\bar{x} = t'$ and $0 \in \mathcal{R}_v^{\beta\eta}$, but $u = \lambda_n \bar{x}.(\lambda_m \bar{y}.cz(c\bar{y}))\bar{x} \in \Psi^c(t, \{0, 1.0\})$ and $u = \lambda_n \bar{x}.(\lambda_m \bar{y}.cz(c\bar{y}))\bar{x} \xrightarrow{1.0}_{\beta\eta} \lambda_n \bar{x}.cz(c\bar{x}) = u'$ and $0 \notin \mathcal{R}_{u'}^{\beta\eta}$.

7.3. Confluence of $\beta\eta$ -developments, hence of $\beta\eta$ -reduction

The next lemma is informative about $\beta\eta$ -developments.

Lemma 7.4. 1. Let $M \in \Lambda$, where $c \notin \text{fv}(M)$, and $\mathcal{F} \subseteq \mathcal{R}_M^{\beta\eta}$. Then:

$$\langle M, \mathcal{F} \rangle \xrightarrow{*}_{\beta\eta d} \langle M', \mathcal{F}' \rangle \iff \exists N \in \Psi^c(M, \mathcal{F}). \exists N' \in \Psi^c(M', \mathcal{F}'). N \xrightarrow{*}_{\beta\eta} N'$$

2. Let $M \in \Lambda$, such that $c \notin \text{fv}(M)$ and $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{R}_M^{\beta\eta}$. If $\langle M, \mathcal{F}_1 \rangle \rightarrow_{\beta\eta d} \langle M', \mathcal{F}'_1 \rangle$ then there exists $\mathcal{F}'_2 \subseteq \mathcal{R}_{M'}^{\beta\eta}$ such that $\mathcal{F}_1 \subseteq \mathcal{F}'_2$ and $\langle M, \mathcal{F}_2 \rangle \rightarrow_{\beta\eta d} \langle M', \mathcal{F}'_2 \rangle$.

Lemma 7.5. (confluence of the $\beta\eta$ -developments)

Let $M \in \Lambda$ such that $c \notin \text{fv}(M)$. If $M \xrightarrow{\mathcal{F}_1}_{\beta\eta d} M_1$ and $M \xrightarrow{\mathcal{F}_2}_{\beta\eta d} M_2$, then there exist $\mathcal{F}'_1 \subseteq \mathcal{R}_{M_1}^{\beta\eta}$, $\mathcal{F}'_2 \subseteq \mathcal{R}_{M_2}^{\beta\eta}$ and $M_3 \in \Lambda$ such that $M_1 \xrightarrow{\mathcal{F}'_1}_{\beta\eta d} M_3$ and $M_2 \xrightarrow{\mathcal{F}'_2}_{\beta\eta d} M_3$.

Notation 7.1. Let $c \notin \text{fv}(M)$. $M \rightarrow_1 M' \iff \exists \mathcal{F}, \mathcal{F}', \langle M, \mathcal{F} \rangle \xrightarrow{*}_{\beta\eta d} \langle M', \mathcal{F}' \rangle$.

The next lemma is needed for the main proof of this section: the Church-Rosser property of the untyped λ -calculus w.r.t. $\beta\eta$ -reduction.

Lemma 7.6. 1. Let $c \notin \text{fv}(M)$. $\forall P \in \Psi^c(M, \emptyset)$. $\mathcal{R}_P^{\beta\eta} = \emptyset$.

2. Let $c \notin \text{fv}(M) \cup \text{fv}(N)$ and $x \neq c$. $\forall P \in \Psi^c(M, \emptyset)$. $\forall Q \in \Psi^c(N, \emptyset)$. $\mathcal{R}_{P[x:=Q]}^{\beta\eta} = \emptyset$.

3. Let $c \notin \text{fv}(M)$. If $p \in \mathcal{R}_M^{\beta\eta}$, $P \in \Psi^c(M, \{p\})$ and $P \rightarrow_{\beta\eta} Q$ then $\mathcal{R}_Q^{\beta\eta} = \emptyset$.

4. Let $c \notin \text{fv}(M)$. If $M \xrightarrow{p}_{\beta\eta} M'$ then $\langle M, \{p\} \rangle \rightarrow_{\beta\eta d} \langle M', \emptyset \rangle$.

5. $\xrightarrow{*}_{\beta\eta} = \xrightarrow{*}_1$.

Finally, here is the main result of this section.

Lemma 7.7. $\Lambda \subseteq \text{CR}^{\beta\eta}$.

8. Conclusion

Reducibility is a powerful method and has been applied to prove using a single method, a number of properties of the λ -calculus (Church-Rosser, strong normalisation, etc.). This paper studied two reducibility methods which exploit the passage from typed (in an intersection type system) to untyped terms. We showed that the first method given by Ghilezan and Likavec [GL02] fails in its aim and we have only been able to provide a partial solution. We adapted the second method given by Koletsos and Stavrinou [KS08] from β to βI -reduction and we generalised it to $\beta\eta$ -reduction. There are differences in the type

systems chosen and the methods of reducibility used by Ghilezan and Likavec on one hand and by Koletsos and Stavrinou on the other. Koletsos and Stavrinou use system \mathcal{D} [Kri90], which has elimination rules for intersection types whereas Ghilezan and Likavec use $\lambda\cap$ and $\lambda\cap^\Omega$ with subtyping. Moreover, Koletsos and Stavrinou's method depends on the inclusion of typable λ -terms in the set of λ -terms possessing the Church-Rosser property, whereas (the working part of) Ghilezan and Likavec's method aims to prove the inclusion of typable terms in an arbitrary subset of the untyped λ -calculus closed by some properties. Moreover, Ghilezan and Likavec consider the $\text{VAR}(\mathcal{P})$, $\text{SAT}(\mathcal{P})$ and $\text{CLO}(\mathcal{P})$ predicates whereas Koletsos and Stavrinou use standard reducibility methods through saturated sets. Koletsos and Stavrinou prove the confluence of developments using the confluence of typable λ -terms in system \mathcal{D} (the authors prove that even a simple type system is sufficient). The advantage of Koletsos and Stavrinou's proof of confluence of developments is that strong normalisation is not needed.

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