Syntactic and Semantic properties of useful \( \lambda \)-calculi: Church-Rosser, reducibility, realisability

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A bit of History
formalisation of Mathematics (and functions)

- First formalisation of the concept of function by Frege (1879, premises of the formalisation of Mathematics)
- Discovery of some paradoxes in Mathematics (around 1900).
- Functions can be applied to any function: reflexiveness.
- Formalisation of the concept of type by Russell to restrict application of functions (1908).
- Formalisation of Mathematics: design of logical systems
Design of the \textit{\lambda-calculus} by Church as part of a formal system for logic and functions (1932).

The full system was inconsistent:

- Church uses the type free \textit{\lambda}-calculus to investigate functions (successful model for computation);
- Church adds simple types (\texttt{int}, \texttt{int} \rightarrow \texttt{int}) to \textit{\lambda}-calculus in a system with logical axioms to deal with logic and function.

Functions are studied as \textit{computational rules} rather than as sets of pairs.
A bit of History
Improvement of the type systems

Definition of typed programming languages such as ML: a statically typed functional programming language based on a polymorphic type system.

Discovery that types in a type system can be associated to formulae in a logical system and that the proofs of formulae can be associated to typable terms: Curry-Howard isomorphism.

- Realisability semantics: connection between recursive functions and intuitionism.
- Reducibility: semantic method based on realisability to prove properties of calculi.
The $\lambda$-calculus, its variants and their properties.

Semantics of typed $\lambda$-calculi using realisability.

Application of the extension of a typed $\lambda$-calculus.

Plan of the thesis.
The λ-calculus, its variants and their properties

The λ-calculus

Let Var be a countably infinite set of variables and \( x, y, z, f \in \text{Var} \).

\[
M, N \in \Lambda := x \mid (\lambda x.M) \mid (MN)
\]

Evaluation:
the \( \beta \)-reduction is the compatible closure\(^\dagger\) of the following rule:

\[
(\lambda x.M_1)M_2 \rightarrow_\beta M_1[x := M_2]
\]

Extensionality:
The \( \eta \)-reduction is the compatible closure of the rule:

\[
\lambda x.Mx \rightarrow_\eta M, \quad \text{if } x \notin \text{fv}(M)
\]

\(^\dagger\)if \( M_1 \rightarrow_\beta M_2 \) then \( \lambda x.M_1 \rightarrow_\beta \lambda x.M_2 \) and \( M_1N \rightarrow_\beta M_2N \) and \( NM_1 \rightarrow_\beta NM_2 \)
For example: the Church-Rosser property.

for all \( M \), if \( M \) reduces to \( M_1 \) and \( M_2 \) then there exists \( M_3 \) such that \( M_1 \) and \( M_2 \) both reduce to \( M_3 \).

Usual steps of a proof of the Church-Rosser property of a \( \lambda \)-calculus:

- Introduction of a new relation (developments).
- Proof of the confluence of this new relation.
- Equivalence between:
  - the transitive closure of the new relation
  - the reflexive and transitive closure of the reduction relation of the considered calculus.
The proofs of the Church-Rosser property can be divided as follows:

▶ First division:
  ▶ Encoding the development using a set of terms: Barendregt et al. (1972), Ghilezan and Kunčak (2001), Koletsos and Stavrinos (2007).

▶ Second division:
  ▶ Using a semantic method: Koletsos and Stavrinos.
  ▶ Using a syntactic method: Barendregt et al., Tait and Martin-Löf, Takahashi.
Our contribution:

- We extended Koletsos and Stavrinos’s semantic proof to the \( \beta\eta \)-case: [KRW08].
- We simplified and extended Koletsos and Stavrinos’s proof as well as that of Ghilezan and Kunčak to obtain a syntactic proof: [KR08].

Our second method is a syntactic proof based on the encoding of developments using sets of terms rather than a reduction relation.

Advantages of our second method:

- We do not deal with types as Koletsos and Stavrinos (or Ghilezan and Kunčak) and our proof is simpler than similar syntactic proofs such as the one of Barendregt et al.
- Our proof of the confluence of developments is parametric (we can easily prove the finiteness of developments).
- Our proof can be seen as a bridge between semantic proofs and syntactic proofs.
Our proof of this property is as follows:

\[ M \xrightarrow{\beta/\beta\eta} \psi(M) \xrightarrow{1/2} P_1 \xrightarrow{c} R_1 \xrightarrow{\beta/\beta\eta} Q_1 \xrightarrow{1/2} Q \]

\[ \psi(P) \xrightarrow{c} R_1 \xrightarrow{\beta/\beta\eta} Q_1 \xrightarrow{c} Q \]

\[ P \xrightarrow{\psi} P_1 \xrightarrow{c} Q_1 \xrightarrow{\psi} Q \]

\[ P_2 \xrightarrow{c} R_1 \xrightarrow{\beta/\beta\eta} Q_1 \xrightarrow{c} Q \]

\[ R \xrightarrow{c} Q_1 \xrightarrow{\beta/\beta\eta} Q \]

\[ \psi(P) \xrightarrow{c} Q_1 \xrightarrow{\psi} Q \]

\[ \psi(Q) \xrightarrow{c} Q \]

\[ \psi(M) \xrightarrow{1/2} P_1 \xrightarrow{c} R_1 \xrightarrow{\beta/\beta\eta} Q_1 \xrightarrow{1/2} Q \]
The λ-calculus, its variants and their properties
The Church-Rosser property - our contribution

Our proof of this property is as follows:

\[ M \xrightarrow{\beta/\beta\eta} \Psi(M) \]

\[ \Psi(P) \rightarrow^{1/2} \Psi(Q) \]

\[ \Psi(P_1) \rightarrow^{1/2} \Psi(Q_1) \]

\[ \Psi(P_2) \rightarrow^{1/2} \Psi(Q_2) \]

\[ \Psi(R_1) \rightarrow^{1/2} \Psi(R) \]

\[ \Psi(R) \rightarrow^{1/2} \Psi(R) \]

\[ \tilde{x} \in \text{Var} \setminus \{c\} \]

\[ \tilde{M} \in \Lambda_{\beta\eta}^{cd} := \tilde{x} \mid \lambda \tilde{x}.\tilde{M} \mid (\lambda \tilde{x}.\tilde{M}_1)\tilde{M}_2 \mid c\tilde{M}_1\tilde{M}_2 \mid c\tilde{M} \]

(Using the simplification of a reducibility method)

Confluence of the simple calculus
Our proof of this property is as follows:

\[ M \xrightarrow{\psi} \Psi(M) \]

\[ \Psi(P) \xrightarrow{\beta/\beta\eta} \Psi(P_1) \]

\[ \Psi(Q) \xrightarrow{\beta/\beta\eta} \Psi(Q_1) \]

\[ \Psi(R) \xrightarrow{\beta/\beta\eta} \Psi(R_1) \]

\[ c \xrightarrow{\beta/\beta\eta} \Psi(c\Psi(P_1)) \]

\[ c \xrightarrow{\beta/\beta\eta} \Psi(c\Psi(Q_1)) \]

\[ (\lambda x.\tilde{M})M_1 \xrightarrow{\beta/\beta\eta} \Psi((\lambda x.\tilde{M})M_1) \]

\[ c\tilde{M}_1 \tilde{M}_2 \xrightarrow{\beta/\beta\eta} \Psi(c\tilde{M}_1 \tilde{M}_2) \]

\[ \bar{x} \in \text{Var} \setminus \{c\} \]

\[ \bar{M} \in \Lambda_{cd}^{\beta\eta} \::= \bar{x} \mid \lambda \bar{x}.\bar{M} \mid (\lambda \bar{x}.\bar{M}_1) \bar{M}_2 \mid c\bar{M}_1 \bar{M}_2 \mid c\bar{M} \]

(Using the simplification of a reducibility method)
Our proof of this property is as follows:

Confluence of developments:

\[ \bar{x} \in \text{Var} \setminus \{c\} \]
\[ \bar{M} \in \Lambda_{\beta\eta}^{\text{cd}} := \bar{x} | \lambda \bar{x}.\bar{M} | (\lambda \bar{x}.\bar{M}_1)\bar{M}_2 | c\bar{M}_1\bar{M}_2 | c\bar{M} \]

(Using the simplification of a reducibility method)
Our proof of this property is as follows:

\[
\begin{align*}
\psi(M) &\xrightarrow{\beta/\beta\eta} P_1 \xrightarrow{c} Q_1 \\
\psi(P) &\xrightarrow{\beta/\beta\eta} R_1 \xrightarrow{c} Q_2 \\
\psi(Q) &\xrightarrow{\beta/\beta\eta} Q \\
\end{align*}
\]

Confluence of developments:

\[
\begin{align*}
\bar{x} &\in \text{Var} \setminus \{c\} \\
\bar{M} &\in \Lambda^\beta_{\text{cd}} ::= \bar{x} | \lambda\bar{x}..\bar{M} | (\lambda\bar{x}..\bar{M}_1)\bar{M}_2 | c\bar{M}_1\bar{M}_2 | c\bar{M} \\
\end{align*}
\]

(using the simplification of a reducibility method)

\[
\begin{align*}
\rightarrow^*_\beta = \rightarrow^*_1 \quad \text{and} \quad \rightarrow^*_{\beta\eta} = \rightarrow^*_2 \\
\text{(Simulation of a reduction by a some developments)}
\end{align*}
\]
Semantics of typed $\lambda$-calculi using realisability

Different ways to cast some lights on a calculus:

- **Denotational semantics**
  - Answer to the question: What terms denote?

- **Operational semantics**
  - Answer to the question: How terms compute?

- **Realisability semantics**
  - Highlight the computational content of a syntactic object.

- ...
Definition of some concepts:

**Principal typing property:** A type system satisfies the principal typing property if for each typable term, there is a typing from which all other typings are obtained via some set of operations.

**Intersection type system:** As the $\forall$ quantifier, intersection types allow to express polymorphism but in a finite way. Intersection types are lists of usages $(\text{int} \rightarrow \text{int} \cap \text{real} \rightarrow \text{real})$.

**Expansion:** Introduced by Coppo, Dezani and Venneri (1980) in order to restore the principal typing property in such systems (extensively improved by Carlier and Wells (2008)).
Semantics of typed $\lambda$-calculi using realizability

Expansion - example

The $\lambda$-term: $M = (\lambda x.x(\lambda y.yz))$

can be assigned the two following typings:

$$\Phi_1 = \langle (z : a) \vdash (((a \rightarrow b) \rightarrow b) \rightarrow c) \rightarrow c \rangle \quad \text{(principal)}$$

$$\Phi_2 = \langle (z : a_1 \sqcap a_2) \vdash (((a_1 \rightarrow b_1) \rightarrow b_1) \sqcap ((a_2 \rightarrow b_2) \rightarrow b_2) \rightarrow c) \rightarrow c \rangle$$

An expansion operation can obtain $\Phi_2$ from $\Phi_1$

In System E (Carlier et al. (2004)), the typing $\Phi_1$ is replaced by:

$$\Phi_3 = \langle (z : ea) \vdash (e((a \rightarrow b) \rightarrow b) \rightarrow c) \rightarrow c \rangle$$

$\Phi_2$ can be obtained from $\Phi_3$ by substituting for $e$ the expansion term:

$$E = (a := a_1, b := b_1) \sqcap (a := a_2, b := b_2)$$
Two steps so far:

- [KNRW08c, KNRW08a]: We provided a complete realisability semantics for an intersection type system with one expansion variable and no expansion mechanism.

- [KNRW08b, KNRW08a]: We provided a complete realisability semantics for an intersection type system with an infinite set of expansion variables and no expansion mechanism.
How do we do that:

- Design of a calculus aiming at the capture of the meaning of an expansion variable: encapsulation of a type.
  - $\lambda$-calculus indexed with natural numbers/list of natural numbers

- Design of a suitable type interpretation.
  - An expansion variable make the realisers change level.

- Proof of the soundness and completeness of the semantics w.r.t. a given type system.
The aim: accurately identify and report the location of some type errors of a piece of code (for a SML-based programming language), by providing a set of minimal and necessary collection of points in the piece of code (a slice).

How does it do that?
- From a piece of code, type error slicing first generates a set of type constraints.
- Then, from this set of type constraints, it runs the enumeration of the set of minimal errors of the piece of code.
- Finally, for each minimal error found, it displays the corresponding slice (parts of the piece of code corresponding to the minimal error).

The current implementation of Type Error slicing is for a larger language than the theory.
Application of the extension of a typed $\lambda$-calculus

Type Error Slicing [HW04]

Our contribution so far:

- Extraction of the different modules of the implementation of Type Error Slicing:
  - Constraint generation
  - Unification
  - Minimisation
  - Enumeration
  - Slicing

  Largely been achieved.

- Proof of the properties of these modules: Correctness and Termination.

  Partially achieved.
During the third year I will focus on the two following subjects:

- **Type error slicing:**
  - Finishing the proofs relative to the current implementation of type error slicing.
  - Extending the Type Error Slicing framework for a rich and sophisticated programming language.
  - Implementing the type error slicer we will develop for the rich language so that we can make our development more practical and can have more impact.

- **Semantics of expansion:** providing a semantics of a typed $\lambda$-calculus with expansion which takes into consideration the expansion mechanism.
Christian Haack and J. B. Wells.
Type error slicing in implicitly typed higher-order languages.

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Challenges and solutions to realizability semantics for intersection types with expansion variables.
Submitted to Fundamenta Informaticae, 2008.

Fairouz Kamareddine, Karim Nour, Vincent Rahli, and Joe B. Wells.
A complete realizability semantics for intersection types and arbitrary expansion variables.

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Developing realizability semantics for intersection types and expansion variables.

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Simplified reducibility proofs of church-rosser for $\beta$- and $\beta\eta$-reduction.
Accepted at LSFA’08, Salvador, Bahia, Brasil, 26 August, 2008.

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