Developing Realisability Semantics for Intersection Types and Expansion Variables

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Abstract

Expansion was invented at the end of the 1970s for calculating *principal typings* for λ-terms in type systems with intersection types. *Expansion variables* (E-variables) were invented at the end of the 1990s to simplify and help mechanize expansion. Recently, E-variables have been further simplified and generalized to also allow calculating other type operators than just intersection. There has been much work on denotational semantics for type systems with intersection types, but none whatsoever before now on type systems with E-variables. Building a semantics for E-variables turns out to be challenging. To simplify the problem, we consider only E-variables, and not the corresponding operation of expansion. We develop a realizability semantics where each use of an E-variable in a type corresponds to an independent level at which evaluation occurs in the λ-term that is assigned the type. In the λ-term being evaluated, the only interaction possible between portions at different levels is that higher level portions can be passed around but never applied to lower level portions. We apply this semantics to two intersection type systems. We show these systems are sound, that completeness does not hold for the first system, and completeness holds for the second system when only one E-variable is allowed (although it can be used many times and nested). As far as we know, this is the first study of a denotational semantics of intersection type systems with E-variables (using realizability or any other approach) and of the difficulties involved.

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1 Introduction

Intersection types were developed in the late 1970s to type λ-terms that are untypable with simple types; they do this by providing a kind of finitary type polymorphism where the usage of types is listed rather than quantified over. They have been useful in reasoning about the semantics of the λ-calculus, and have been investigated for use in static program analysis. Coppo, Dezani, and Venneri [4] introduced the operation of *expansion* on *typings* (pairs of a type environment and a result type) for calculating the possible typings of a term when using intersection types. Expansion is a crucial part of a procedure for calculating *principal typings* and thus helps support compositional type inference. As a simple example, the λ-term $M = (\lambda x.x(\lambda y.yz))$ can be assigned the typing $\Phi_1 = \langle (z : a) \vdash (((a{\to}b){\to}b){\to}c){\to}c \rangle$, which happens to be its principal typing. The term M can also be assigned the typing $\Phi_2 = \langle (z : a_1{\sqcap}a_2) \vdash (((a_1{\to}b_1){\to}b_1){\sqcap}((a_2{\to}b_2){\to}b_2){\to}c){\to}c \rangle$, and an expansion operation can obtain Φ_2 from Φ_1. Because the early definitions of expansion were complicated, E-variables were introduced in order to make the calculations easier to mechanize and reason about. For example, in System E [2], the typing Φ_1 from above is replaced by $\Phi_3 = \langle (z : ea) \vdash (e((a \to b) \to b) \to c) \rangle$, which differs from Φ_1 by the insertion of the E-variable e at two places, and Φ_2 can be obtained from Φ_3 by substituting for e the *expansion term* $E = (a := a_1, b := b_1) \sqcap (a := a_2, b := b_2)$. Carlier and Wells [3] have surveyed the history of expansion and also E-variables.

Various kinds of denotational semantics have helped in reasoning about the properties of entire type systems and of specific typed terms. E-variables pose serious challenges for semantics. Most commonly, a type's semantics is given as a set of closed λ-terms with behavior related to the specification given by the type. In many kinds of semantics, the meaning of a type T is calculated by an expression $[T]_\nu$ that takes two parameters, the type T and also a valuation ν that assigns to type variables the same kind of meanings that are assigned to types. To extend this idea to types with E-variables, we would need to devise some space of possible meanings for E-variables. Given that a type $e\,T$ can be turned by expansion into a new type $S_1(T) \sqcap S_2(T)$, where S_1 and S_2 are arbitrary substitutions (they can be arbitrary further expansions), and that this can introduce an unbounded number of new variables (both E-variables and regular type variables), the situation is complicated.

Because it is unclear how to devise a space of meanings for expansions and E-variables, we instead develop a space of meanings for types that is hierarchical in the sense of having many levels. When assigning meanings to types, we make each use of E-variables simply change levels. We specifically avoid trying to give a semantics to the operation of expansion, and instead treat only the E-variables. Although this idea is not perfect, it seems to go quite far in giving an intuition for E-variables, namely that each E-variable acts as a kind of capsule that isolates parts of the λ-term being analyzed by the typing. Parts of the λ-term that are typed inside the uses of the E-variable-introduction typing rule for a particular E-variable e can interact with each other, and parts outside e can only pass the parts inside e around. The E-variable e of course also shows up in the types, and isolates the portions of the types contributed by the portions of the term inside the relevant uses of E-variable-introduction.

The semantic approach we use is realisability semantics. Atomic types (e.g., type variables) are interpreted as sets of λ-terms that are *saturated*, meaning in this case that they are closed under β-expansion (i.e., β-reduction in reverse). Arrow and intersection types are interpreted naturally by function spaces and set intersection. Realisability allows showing *soundness* in the sense that the meaning of a type T contains all closed λ-terms that can be assigned T as their result type. This has been shown useful for characterizing the behavior of typed λ-terms [12]. One also wants to show the converse of soundness which is called *completeness*, i.e., that every closed λ-term in the meaning of T can be assigned T as its result type.

Hindley [8, 9, 10] was the first to study this notion of completeness for a simple type system and he showed that all the types of that system have the completeness property. Then, he generalised his completeness proof for an intersection type system [7]. Using his completeness theorem for the realisability semantics based on the sets of λ-terms saturated by $\beta\eta$-equivalence, Hindley has shown that simple types are uniquely realised by the λ-terms which are typable by these types. However, Hindley's result does not hold for his intersection type system and the completeness theorems were established with the sets of λ-terms saturated by $\beta\eta$-equivalence. In this paper, our completeness result depends instead only on the weaker requirement of β-equivalence, and we have managed to make simpler proofs that avoid needing η-reduction, confluence (a.k.a. Church/Rosser), or strong normalisation (SN) (although we do establish both confluence and SN for both β and $\beta\eta$).

Other work on realizability we have consulted includes that by Labib-Sami [13], Farkh and Nour [6], and Coquand [5], although none of this work deals with intersection types or E-variables. Related work on realisability that deals with intersection types includes that by Kamareddine and Nour [11], which gives a realisability semantics with soundness and completeness for an intersection type system. This system is quite different from the ones in this paper, because it allows the universal type ω. We do not currently know how to build a semantics that supports both ω and E-variables. The method of levels we use in this paper would need to assign ω to every level, which is impossible. Further work will be needed on this point.

In this paper we study the λI-calculus typed with two representative intersection type systems. The restriction to λI (where in every $(\lambda x.M)$ the variable x must appear free in M) is motivated by not knowing how to support the ω type. For one of these systems, we show that subject reduction and hence completeness do not hold whereas for the second system, subject reduction holds and completeness will hold if at most one single E-variable is used. As far as we know, this is the first study of a denotational semantics of intersection type systems with E-variables (using realizability or any other approach) and of the difficulties involved.

Section 2 introduces the $\lambda I^{\mathbb{N}}$-calculus, which is the λI-calculus with each variable marked by a natural number *degree*. Section 3 introduces the syntax and terminology for types, and also the realisability semantics. Section 4 introduces our two intersection type systems with E-variables where in one, the syntax of types is not restricted but in the other it is restricted but then extended with a subtyping relation. We show that subject reduction (SR) and completeness do not hold for the first system, and that SR holds for the second system. In section 5 we show the soundness of the realisability semantics for both systems and give a number of examples. Section 6 shows completeness does not hold for the second system if more than one expansion variable is used, but does hold for a restriction of this system to one single E-variable. This is an important study in the semantics of intersection type systems with expansion variables since a unique expansion variable can be used many times and can occur nested. Section 7 concludes. In the appendices we establish confluence and strong normalisation results as well as results related to the usual unindexed λI-calculus.

2 The pure $\lambda I^{\mathbb{N}}$-calculus

In this section we give an indexed version of the λI-calculus where indices (which range over the set of natural numbers $\mathbb{N}$) help categorise the so-called good terms (where the degree (or level) of a function is always smaller than that of its arguments). This amounts to having the full λI-calculus at each level (index) and creating new λI-terms through a mixing recipe.

We assume that if a metavariable v ranges over a set $\mathcal{S}$ then v_i for $i \geq 0$ and v', v'', etc. also range over $\mathcal{S}$.

Definition 1 *1. Let $\mathcal{V}$ be a denumerably infinite set of variables. The set of terms $\mathcal{M}$, the set of good terms $\mathbb{M} \subset \mathcal{M}$, the set of free variables $FV(M)$ of a term in $M \in \mathcal{M}$, the degree $d(M)$ of a term M and the joinability $M \diamond N$ of terms M and N are defined by simultaneous induction:*

- *If $x \in \mathcal{V}$ and $n \in \mathbb{N}$, then $x^n \in \mathcal{M} \cap \mathbb{M}$, $FV(x^n) = \{x^n\}$, and $d(x^n) = n$.*
- *If $M, N \in \mathcal{M}$ such that $M \diamond N$ (see below), then*
  - *$(M\, N) \in \mathcal{M}$, $FV((M\, N)) = FV(M) \cup FV(N)$ and $d((M\, N)) = \min(d(M), d(N))$ (where $\min$ is the minimum)*
  - *If $M \in \mathbb{M}$, $N \in \mathbb{M}$ and $d(M) \leq d(N)$ then $(M\, N) \in \mathbb{M}$.*
- *If $M \in \mathcal{M}$ and $x^n \in FV(M)$, then*
  - *$(\lambda x^n.M) \in \mathcal{M}$, $FV((\lambda x^n.M)) = FV(M) \setminus \{x^n\}$, and $d(\lambda x^n.M_1) = d(M_1)$.*
  - *If $M \in \mathbb{M}$ then $\lambda x^n.M \in \mathbb{M}$.*

2.
- *Let $M, N \in \mathcal{M}$. We say that M and N are joinable and write $M \diamond N$ iff $\forall x \in \mathcal{V}$, if $x^m \in FV(M)$ and $x^n \in FV(N)$, then $m = n$.*
- *If $\mathcal{X} \subseteq \mathcal{M}$ such that $\forall M, N \in \mathcal{X}, M \diamond N$, we write, $\diamond\mathcal{X}$.*
- *If $\mathcal{X} \subseteq \mathcal{M}$ and $M \in \mathcal{M}$ such that $\forall N \in \mathcal{X}, M \diamond N$, we write, $M \diamond \mathcal{X}$.*

The $\diamond$ property ensures that in any term M, variables have unique degrees.

We assume the usual definition ([1, 12]) of subterms and the usual convention for parentheses and their omittance. Note that every subterm of $M \in \mathcal{M}$ (resp. $\mathbb{M}$) is also in $\mathcal{M}$ (resp. $\mathbb{M}$). We let $x, y, z,$ etc. range over $\mathcal{V}$ and $M, N, P, M_1, M_2, \ldots$ range over $\mathcal{M}$ and use $=$ for syntactic equality.

3. For each $n \in \mathbb{N}$, we let:

$\mathcal{M}^n = \{M \in \mathcal{M} \;/\; d(M) = n\}$ $\qquad$ $\mathcal{M}^{>n} = \mathcal{M}^{\geq n+1}$

$\mathcal{M}^{\geq n} = \{M \in \mathcal{M} \;/\; d(M) \geq n\}$ $\qquad$ $\mathbb{M}^n = \mathbb{M} \cap \mathcal{M}^n$

4. The usual substitution $M[x^m := N]$ of $N \in \mathcal{M}$ for all free occurrences of x^m in $M \in \mathcal{M}$ only matters when $M \diamond N$. For $n \geq 0$, $M[(x_i^{n_i} := N_i)_{1 \leq i \leq n}]$ (or simply $M[(x_i^{n_i} := N_i)_n]$), the simultaneous substitution of N_i for all free occurrences of $x_i^{n_i}$ in M only matters when $\diamond\mathcal{X}$ where $\mathcal{X} = \{M\} \cup \{N_i / 1 \leq i \leq n\} \subseteq \mathcal{M}$. Hence we restrict substitution accordingly to incorporate the $\diamond$ condition. With $\mathcal{X}$ as above, $M[(x_i^{n_i} := N_i)_n]$ is only defined when $\diamond\mathcal{X}$. We write $M[(x_i^{n_i} := N_i)_{1 \leq i \leq 1}]$ as $M[x_1^{n_1} := N_1]$ obviously.

5. On $\mathcal{M}$, we take terms modulo α-conversion given by:

$$\lambda x^n.M = \lambda y^n.(M[x^n := y^n]) \text{ where } \forall m, y^m \not\in FV(M)$$

Note here that $y^n \diamond M$ and hence, $M[x^n := y^n] \in \mathcal{M}$ and since $x^n \in FV(M)$ then $y^n \in FV(M[x^n := y^n])$. Hence, if $\lambda x^n.M \in \mathcal{M}$, $y^n \not\in FV(M)$ and $y^n \diamond M$ then $\lambda y^n.(M[x^n := y^n]) \in \mathcal{M}$. Note also that:

$$\text{If } x \neq y, \text{ then } \forall\, i, j \in \mathbb{N},\ x^i \neq y^j. \text{ Moreover, } x^i = x^j \text{ iff } i = j.$$

We use the Barendregt convention (BC) where the names of bound variables differ from the free ones and where we rewrite terms so that not both λx^n and λx^m co-occur when $n \neq m$. BC ensures that:

- *If $M[(x_i^{n_i} := N_i)_n] \in \mathcal{M}$ then $\forall m_i, \forall 1 \leq i \leq n$, $\lambda x_i^{m_i}$ does not occur in M.*
- *If $M, N \in \mathcal{M}$, $M \diamond N$ and $x^n \in FV(M)$ then $\forall m$, λx^m does not occur in N.*

The next lemma states when an application or an abstraction term is good.

Lemma 2 *1. (M is good and $x^n \in FV(M)$) iff $\lambda x^n.M$ is good.*

2. (M_1 and M_2 are good, $M_1 \diamond M_2$ and $d(M_1) \leq d(M_2)$) iff $M_1 M_2$ is good.

Proof The only if direction is by definition. The if direction, for each of 1. and 2. is by cases on the derivation $\lambda x^n.M$ is good respectively $M_1 M_2$ is good. □

Since only joinable terms matter, the next lemma states results about $\diamond$.

Lemma 3 *1. On $\mathcal{M}$, $\diamond$ is reflexive and symmetric but not transitive.*

2. Let $M, N, M', N' \in \mathcal{M}$ such that M' is a subterm of M and N' is a subterm of N. If $M \diamond N$, then $M' \diamond N'$.

3. (a) Let $M, (N_1 N_2) \in \mathcal{M}$. We have $M \diamond \{N_1, N_2\}$ iff $M \diamond (N_1 N_2)$.

(b) Let $M, \lambda x^n.N \in \mathcal{M}$. We have $M \diamond N$, iff $M \diamond (\lambda x^n.N)$.

(c) Let $M, N[(x_i^{n_i} := N_i)_p] \in \mathcal{M}$ and $\mathcal{X} = \{N\} \cup \{N_i / 1 \leq i \leq p\} \subset \mathcal{M}$.

- *If $M \diamond \mathcal{X}$, then $M \diamond N[(x_i^{n_i} := N_i)_p]$.*

4. Let $M_1[(x_i^{n_i} := N_i)_p] \in \mathcal{M}$, $M_2[(x_i^{n_i} := N_i)_p] \in \mathcal{M}$ and $\mathcal{X} = \{M_1, M_2\} \cup \{N_i / 1 \leq i \leq p\}$. We have: If $\diamond\mathcal{X}$ then $M_1[(x_i^{n_i} := N_i)_p] \diamond M_2[(x_i^{n_i} := N_i)_p]$.

5. Let $M \in \mathcal{M}$. We have: $d(M) = \min(n \in \mathbb{N} \ / \ x^n$ occurs in $M)$.

6. Let $\mathcal{X} = \{M\} \cup \{N_i / 1 \leq i \leq p\} \subset \mathcal{M}$. We have:

(a) $\diamond\mathcal{X}$ iff $M[(x_i^{n_i} := N_i)_p] \in \mathcal{M}$.

(b) If $\diamond\mathcal{X}$ and $\forall\ 1 \leq i \leq p$, $d(N_i) = n_i$, then $d(M[(x_i^{n_i} := N_i)_p]) = d(M)$.

7. Let $M, N, P \in \mathcal{M}$. If $\diamond\{M, N, P\}$, $y \neq x$ and $x^n \notin FV(P)$, then $M[x^n := N][y^m := P] = M[y^m := P][x^n := N[y^m := P]]$.

8. Let $M, N, P \in \mathcal{M}$. If $M \diamond P$ and $FV(M) = FV(N)$ then $N \diamond P$.

9. Let $M, N \in \mathcal{M}$ where $d(N) = n$ and $x^n \in FV(M)$. We have: $M[x^n := N]$ is good iff M and N are good and $M \diamond N$.

Proof

1. For reflexivity, we show by induction on $M \in \mathcal{M}$ that if $x^n, x^m \in FV(M)$, then $n = m$. Symmetry is by definition of $\diamond$. For failure of transitivity take z^2, y^1 and z^3.

2. Let $x^m \in FV(M')$ and $x^n \in FV(N')$. If $x^m \in FV(M)$ and $x^n \in FV(N)$ use $M \diamond N$. The cases a) $x^m \in FV(M)$ and λx^n occurs in N and b) λx^m occurs in M and $x^n \in FV(N)$ are not possible by BC. Finally, if λx^m occurs in M and λx^n occurs in N, then by BC, $n = m$.

3. Simple check of the $\diamond$ condition using 2.

4. By 3c, $M_1 \diamond M_2[(x_i^{n_i} := N_i)_p]$ and $N_j \diamond M_2[(x_i^{n_i} := N_i)_p]$ $\forall\ 1 \leq j \leq p$, and, by 3c again, $M_1[(x_i^{n_i} := N_i)_p] \diamond M_2[(x_i^{n_i} := N_i)_p]$.

5. By induction on M.

6. 6a is by definition of substitution. 6b is by induction on M.

7. By induction on M using 3c and 6a.

8. If $x^n \in FV(N) = FV(M)$ and $x^p \in FV(P)$ then since $M \diamond P$, $n = p$.

9. By induction on M.

   – By definition of substitution, $x^n[x^n := N]$ is good iff x^n and N are good and $x^n \diamond N$.

   – $(\lambda y^m.M')[x^n := N]$ is good $\Leftrightarrow$ $\lambda y^m.M'[x^n := N]$ is good and $y^m \in FV(M') \setminus FV(N)$ (since $\lambda y^m.M' \in \mathcal{M}$ using BC) $\Leftrightarrow^{lemma\ 2}$ $M'[x^n := N]$ is good, $y^m \in FV(M'[x^n := N])$ and $y^m \in FV(M') \setminus FV(N)$ $\Leftrightarrow^{IH}$ M' and N are good, $M' \diamond N$, $y^m \in FV(M'[x^n := N])$ and $y^m \in FV(M') \setminus FV(N)$ $\Leftrightarrow^{3b\&lemma\ 2}$ $\lambda y^m.M'$ and N are good and $\lambda y^m.M' \diamond N$.

   – $(M_1M_2)[x^n := N]$ is good $\Leftrightarrow$ $M_1[x^n := N]M_2[x^n := N]$ is good and $\diamond\{M_1, M_2, N\}$ (since $(M_1M_2)[x^n := N] \in \mathcal{M}$) $\Leftrightarrow^{6b\&lemma\ 2}$ $M_1[x^n := N]$ and $M_2[x^n := N]$ are good, $M_1[x^n := N] \diamond M_2[x^n := N]$, $\diamond\{M_1, M_2, N\}$ and $\mathrm{d}(M_1) = \mathrm{d}(M_1[x^n := N]) \leq \mathrm{d}(M_2[x^n := N]) = \mathrm{d}(M_2)$ $\Leftrightarrow^{IH}$ M_1, M_2 and N are good, $\diamond\{M_1, M_2, N\}$ and $\mathrm{d}(M_1) \leq \mathrm{d}(M_2)$ $\Leftrightarrow^{3a\&lemma\ 2}$ M_1, M_2 and N are good and $(M_1M_2) \diamond N$. □

Now we define the beta reduction relation on the $\lambda I^{\mathbb{N}}$-calculus.

Definition 4 *1. A relation R on $\mathcal{M}$ is compatible iff for all $M, N, P \in \mathcal{M}$:*

- *If MRN and $x^n \in FV(M) \cap FV(N)$, then $(\lambda x^n.M)R(\lambda x^n.N)$.*
- *If MRN, $M \diamond P$ and $N \diamond P$, then $(MP)R(NP)$ and $(PM)R(PN)$.*

2. The reduction relation $\rhd_\beta$ on $\mathcal{M}$ is defined as the least compatible relation closed under the rule: $(\lambda x^n.M)N \rhd_\beta M[x^n := N]$ if $d(N) = n$.

3. We denote by $\rhd_\beta^*$ the reflexive and transitive closure of $\rhd_\beta$. We also denote by $\simeq_\beta$ the equivalence relation induced by $\rhd_\beta^*$.

The next lemma shows that beta reduction is well defined on the $\lambda I^{\mathbb{N}}$-calculus.

Lemma 5 *$\rhd_\beta$ is a well defined relation on $\mathcal{M}$. I.e., if $M \in \mathcal{M}$ and $M \rhd_\beta N$ then $N \in \mathcal{M}$. Hence, $\rhd_\beta^*$ is also a well defined relation on $\mathcal{M}$.*

Proof By induction on $M \rhd_\beta N$. We only treat the basic case. Let $(\lambda x^n.M_1)M_2 \rhd_\beta M_1[x^n := M_2]$ where $\mathrm{d}(M_2) = n$. By Lemmas 3.2 and 3.6a, $M_1[x^n := M_2] \in \mathcal{M}$ and $M_1 \diamond M_2$. We show by induction on $M \rhd_\beta^* N$ that if $M \in \mathcal{M}$ and $M \rhd_\beta^* N$ then $N \in \mathcal{M}$. □

The next lemma shows that the beta reduction relations preserves the free variables, degrees and goodness of terms.

Lemma 6 *Let $M, N \in \mathcal{M}$ such that $M \rhd_\beta^* N$. We have:*

1. $FV(M) = FV(N)$ and $d(M) = d(N)$.

2. M is good iff N is good.

Proof 1. By induction on the derivation of $M \rhd_\beta^* N$. We only treat the following:

- Assume $(\lambda x^n.M_1)M_2 \rhd_\beta M_1[x^n := M_2]$ where $\mathrm{d}(M_2) = n$. Since $\mathrm{d}(M_2) = n$, then, by Lemmas 3.6b, $\mathrm{d}(M_1[x^n := M_2]) = \mathrm{d}(M_1) = \mathrm{d}(\lambda x^n.M_1) = \mathrm{d}((\lambda x^n.M_1)M_2)$. Also, $FV((\lambda x^n.M_1)M_2) = (FV(M_1) \setminus \{x^n\}) \cup FV(M_2) = FV(M_1[x^n := M_2])$.

2. By induction on the length of the derivation $M \rhd_\beta^* N$.

- If the length of the derivation is 0, nothing to prove.

- Case $M \rhd_\beta N$. We do the proof by induction on the derivation of $M \rhd_\beta N$.
  - Let $M = (\lambda x^n.M_1)M_2 \rhd_\beta M_1[x^n := M_2] = N$ with $\mathrm{d}(M_2) = n$. $M \in \mathcal{M}$, so $x^n \in FV(M_1)$ and $(\lambda x^n.M_1) \diamond M_2$. Since $(\lambda x^n.M_1) \diamond M_2$, by lemma 3, $M_1 \diamond M_2$.
    - Assume N is good. By lemma 3.9, M_1 and M_2 are good. Since M_1 is good and $x^n \in FV(M_1)$, by definition $\lambda x^n.M_1$ is good. Since $x^n \in FV(M_1)$, by lemma 3.5, $\mathrm{d}(M_1) \leq n$, so $\mathrm{d}(\lambda x^n.M_1) = \mathrm{d}(M_1) \leq n = \mathrm{d}(M_2)$. So, by definition M is good.
    - Assume M is good. Then, by lemma 2, $\lambda x^n.M_1$ and M_2 are good, and $\mathrm{d}(\lambda x^n.M_1) \leq \mathrm{d}(M_2)$. Since $(\lambda x^n.M_1) \diamond M_2$, by lemma 3.2, $M_1 \diamond M_2$. By lemma 2 and since $\lambda x^n.M_1$ is good, M_1 is good. Since M_1 and M_2 are good, $x^n \in FV(M_1)$ and $\mathrm{d}(M_2) = n$, by lemma 3.9, N is good.
  - Let $M = PQ \rhd_\beta PQ' = N$ and $Q \rhd_\beta Q'$, $P \diamond Q$ and $P \diamond Q'$. Since $Q \rhd_\beta Q'$, by lemma 6.1, $\mathrm{d}(Q) = \mathrm{d}(Q')$.
    - If N is good, then by lemma 2, P and Q' are good and $\mathrm{d}(P) \leq \mathrm{d}(Q')$. Hence, $\mathrm{d}(P) \leq \mathrm{d}(Q)$. Moreover, by IH, Q is good. Hence, by definition, M is good.
    - If M is good then by lemma 2, P and Q are good and $\mathrm{d}(P) \leq \mathrm{d}(Q)$. Hence, $\mathrm{d}(P) \leq \mathrm{d}(Q')$. Moreover, by IH, Q' is good. Hence, by definition, N is good.
  - Let $M = PQ \rhd_\beta P'Q = N$ and $P \rhd_\beta P'$, $P \diamond Q$ and $P' \diamond Q$. The proof is similar to the previous item.
  - Let $M = \lambda y^m.M' \rhd_\beta \lambda y^m.N' = N$, $y^m \in FV(M') \cap FV(N')$, and $M' \rhd_\beta N'$.
    - If N is good, then by lemma 2, N' is good and by IH, M' is good. Hence, by definition, M is good.
    - If M is good then by lemma 2, M' is good and by IH, N' is good. Hence, by definition, N is good.
- Case $M \rhd_\beta N_1 \rhd_\beta^* N$ use IH. □

The next definition turns terms of degree n into terms of higher degrees and also, if $n > 0$, they can be turned into terms of lower degrees.

Definition 7 *1. We define* $^+ : \mathcal{M} \mapsto \mathcal{M}$ *and* $^- : \mathcal{M}^{>0} \mapsto \mathcal{M}$ *by:*

$(x^n)^+ = x^{n+1}$	$(x^n)^- = x^{n-1}$
$(M_1\ M_2)^+ = M_1^+\ M_2^+$	$(M_1\ M_2)^- = M_1^-\ M_2^-$
$(\lambda x^n.M)^+ = \lambda x^{n+1}.M^+$	$(\lambda x^n.M)^- = \lambda x^{n-1}.M^-$

2. Let $\mathcal{X} \subseteq \mathcal{M}$*. If* $\forall M \in \mathcal{X}$*,* $d(M) > 0$*, we write* $d(\mathcal{X}) > 0$*. We define:* $\mathcal{X}^+ = \{M^+ \ / \ M \in \mathcal{X}\}$ *If* $d(\mathcal{X}) > 0$*,* $\mathcal{X}^- = \{M^- \ / \ M \in \mathcal{X}\}$*.*

3. If $d(M) \geq n > 0$*, we write* M^{-n} *for* $(\dots(M\overbrace{^-)^- \dots ^-)^-}^{n}$*.*
It is easy to show that M^{-n} *is well defined.*

The next lemma shows that the lifting of a term to higher or lower degrees, is a well behaved operation with respect to all that matters (free variables, reduction, joinability, substitution, etc.).

Lemma 8 *Let* $\succ \in \{\rhd, \rhd^*\}$*,* $p \geq 0$ *and* $M, N, N_1, N_2, \dots, N_p \in \mathcal{M}$*.*

1. *(a)* $d(M^+) = d(M)+1$*,* $(M^+)^- = M$ *and* $x^n \in FV(M^+)$ *iff* $x^{n-1} \in FV(M)$*.*

   (b) *If* $d(M) > 0$*, then* $M^- \in \mathcal{M}$*,* $d(M^-) = d(M) - 1$*,* $(M^-)^+ = M$ *and* $x^n \in FV(M^-)$ *iff* $x^{n+1} \in FV(M)$*.*

   (c) *Let* $\mathcal{X} \subset \mathcal{M}$*. Then,*

      i. $\diamond\mathcal{X}$ *iff* $\diamond\mathcal{X}^+$*.*

      ii. *If* $d(\mathcal{X}) > 0$ *then* $\diamond\mathcal{X}$ *iff* $\diamond\mathcal{X}^-$*.*

      iii. $M \in \mathcal{X}^+$ *iff (*$M^- \in \mathcal{X}$ *and* $d(M) > 0$*).*

   (d) M *is good iff* M^+ *is good.*

   (e) *If* $d(M) > 0$ *then* M *is good iff* M^- *is good.*

2. *Let* $\mathcal{X} = \{M\} \cup \{N_i / 1 \leq i \leq p\} \subset \mathcal{M}$*.*
   If $\diamond\mathcal{X}$*, then* $(M[(x_i^{n_i} := N_i)_p])^+ = M^+[(x_i^{n_i+1} := N_i^+)_p]$*.*

3. *If* $d(M), d(N) > 0$*, and* $M \diamond N$*, then* $(M[x^{n+1} := N])^- = M^-[x^n := N^-]$*.*

4. *If* $M \succ_\beta N$*, then* $M^+ \succ_\beta N^+$*.*

5. *If* $d(M) > 0$ *and* $M \succ_\beta N$*, then* $M^- \succ_\beta N^-$*.*

6. *If* $M \succ_\beta N^+$*, then* $M^- \succ_\beta N$*.*

7. *If* $M^+ \succ_\beta N$*, then* $M \succ_\beta N^-$*.*

8. *Let* $P \in \mathcal{M}$*. If* $M \succ_\beta N$*,* $P \succ_\beta Q$ *and* $M \diamond P$*, then* $N \diamond Q$*.*

9. *If* $M \succ_\beta N$*,* $M \diamond P$ *and* $d(P) = n$*, then* $M[x^n := P] \succ_\beta N[x^n := P]$*.*

10. *If* $N \succ_\beta P$ *and* $M \diamond N$*, then* $M[x^n := N] \rhd_\beta^* M[x^n := P]$*.*

11. *If* $M \rhd_\beta^* N$*,* $P \rhd_\beta^* P'$*,* $M \diamond P$ *and* $d(P) = n$*, then* $M[x^n := P] \rhd_\beta^* N[x^n := P']$*.*

Proof

1. 1a and 1b are by induction on M. For 1(c)i use 1a. For 1(c)ii use 1b. As to 1(c)iii, if $M \in \mathcal{X}^+$, then $M = P^+$ where $P \in X$ and by 1a, $\mathrm{d}(M) = \mathrm{d}(P) + 1 > 0$ and $M^- = (P^+)^- = P$. Hence, $M^- \in \mathcal{X}$ and $\mathrm{d}(M) > 0$. On the other hand, if $M^- \in \mathcal{X}$ and $\mathrm{d}(M) > 0$ then by 1b, $M = P^+$ and $(M^-)^+ = M \in \mathcal{X}^+$. Moreover, 1d is by induction on M using 1a, 1(c)i and lemma 2. Finally, for 1e, by 1b and 1d, $M = (M^-)^+ \in \mathbb{M} \Leftrightarrow M^- \in \mathbb{M}$.

2. By induction on M (by 1(c)i and lemma 3.6, we have $M[(x_i^{n_i} := N_i)_p] \in \mathcal{M}$ and $M^+[(x_i^{n_i+1} := N_i^+)_p] \in \mathcal{M}$).

3. By induction on M (by 1(c)ii and lemma 3.6, we have $M[x^{n+1} := N] \in \mathcal{M}$ and $M^-[x^n := N^-] \in \mathcal{M}$).

4. The case $\succ = \rhd$ is by induction on $M \rhd_\beta N$ using 1. and 2., case $\rhd_\beta^*$ is by induction on the length of $M \rhd_\beta^* N$ using the result for case $\rhd_\beta$.

5. Similar to 4.

6. By lemma 6.1, 1a and 5, $M^- \succ N$.

7. Similar to 6.

8. Note that, by lemma 6.1, $FV(M) = FV(N)$ and $FV(P) = FV(Q)$.

9. Case $\succ = \rhd$ is by induction on M using lemmas 3.6b and 3.7. Case $\rhd_\beta^*$ is by induction on the length of $M \rhd_\beta^* N$ using the result for case $\rhd_\beta$.

10. Case $\succ = \rhd$ is by induction on M. Case $\rhd_\beta^*$ is by induction on the length of $M \rhd_\beta^* N$ using the result for case $\rhd_\beta$.

11. Use 9 and 10. □

Normal forms are defined as usual.

Definition 9 *1. We say that $M \in \mathcal{M}$ is in β-normal form (or simply is in normal form) if there is no $N \in \mathcal{M}$ such that $M \rhd_\beta N$.*

2. We say that $M \in \mathcal{M}$ is β-normalising (or simply normalising) if there is an $N \in \mathcal{M}$ such that $M \rhd^*_\beta N$ and N is in normal form.

Next we give a lemma that will be used in the rest of the article.

Lemma 10 *1. If $M[y^n := x^n] \rhd_\beta N$ then $M \rhd_\beta N'$ where $N = N'[y^n := x^n]$.*

2. If $M[y^n := x^n]$ has a β-normal form then M has a β-normal form.

3. Let $k \geq 1$. If $Mx_1^{n_1}...x_k^{n_k}$ is normalizing, then M is normalizing.

4. Let $k \geq 1$, $1 \leq i \leq k$, $l \geq 0$, $x_i^{n_i}N_1...N_l$ be in normal form and M be closed. If $Mx_1^{n_1}...x_k^{n_k} \rhd^*_\beta x_i^{n_i}N_1...N_l$, then for some $m \geq i$ and $n \leq l$, $M \rhd^*_\beta \lambda x_1^{n_1}....\lambda x_m^{n_m}.x_i^{n_i}M_1...M_n$ where $n+k = m+l$, $M_j \simeq_\beta N_j$ for every $1 \leq j \leq n$ and $N_{n+j} \simeq_\beta x_{m+j}^{n_{m+j}}$ for every $1 \leq j \leq k-m$.

Proof

1. By induction on $M[y^n := x^n] \rhd_\beta N$.

2. $M[y^n := x^n] \rhd^*_\beta P$ where P is in β-normal form. The proof is by induction on $M[y^n := x^n] \rhd^*_\beta P$ using 1.

3. By induction on $k \geq 1$. We only prove the basic case. The proof is by cases.
   - If $M\, x_1^{n_1} \rhd^*_\beta M'\, x_1^{n_1}$ where $M'\, x_1^{n_1}$ is in β-normal form and $M \rhd^*_\beta M'$ then M' is in β-normal form and M is β-normalising.
   - If $M\, x_1^{n_1} \rhd^*_\beta (\lambda y^{n_1}.N)\, x_1^{n_1} \rhd_\beta N[y^{n_1} := x_1^{n_1}] \rhd^*_\beta P$ where P is in β-normal form and $M \rhd^*_\beta \lambda y^{n_1}.N$ then by 2, N has a β-normal form and so, $\lambda y^{n_1}.N$ has a β-normal form. Hence, M has a β-normal form.

4. By 3, M is normalizing, and, since M is closed, its normal form is $\lambda x_1^{n_1}....\lambda x_m^{n_m}.z^r M_1...M_n$ for $n, m \geq 0$.
   Since by theorem 68, $x_i^{n_i}N_1...N_l \simeq_\beta (\lambda x_1^{n_1}....\lambda x_m^{n_m}.z^r M_1...M_n)x_1^{n_1}...x_k^{n_k}$ then $m \leq k$, $x_i^{n_i}\ N_1...N_l \simeq_\beta z^r M_1...M_n x_{m+1}^{n_{m+1}}...x_k^{n_k}$. Hence, $z^r = x_i^{n_i}$, $n \leq l$, $i \leq m$, $l = n + (k-(m+1)) + 1 = n+k-m$, $M_j \simeq_\beta N_j$ for every $1 \leq j \leq n$ and $N_{n+j} \simeq_\beta x_{m+j}^{n_{m+j}}$ for every $1 \leq j \leq k-m$.

□

3 The types and their realisability semantics

In this section, we introduce the basic sets of types we use in our type systems and the notions of a degree of a type and of a good type. We also introduce the realisability semantics where good types can only contain good terms.

3.1 The types

This paper studies two type systems. In the first system, there are no restrictions on where the arrow occurs. In the second, arrows cannot occur to the left of intersections or expansions. The next definition introduces these two basic sets of types and the notions of a degree of a type and of a good type.

Definition 11 (Types, good types, degree of a type)

1. *Assume two denumerably infinite sets $\mathcal{A}$ (of atomic types) and $\mathcal{E}$ (of expansion variables). Let $a, b, c, a_1, \ldots$ range over $\mathcal{A}$ and e range over $\mathcal{E}$.*

2. *The set of types $\mathcal{T}$ is defined by: $\mathcal{T} ::= \mathcal{A} \mid \mathcal{T} \to \mathcal{T} \mid \mathcal{T} \sqcap \mathcal{T} \mid \mathcal{E}\mathcal{T}$.*

3. *The set of types $\mathbb{U}$ is defined by:*

$$\mathbb{U} ::= \mathbb{U} \sqcap \mathbb{U} \;\mid\; \mathcal{E}\mathbb{U} \;\mid\; \mathbb{T} \qquad\qquad \text{where } \mathbb{T} ::= \;\mathcal{A} \;\mid\; \mathbb{U} \to \mathbb{T}$$

   Note that $\mathbb{T} \subseteq \mathbb{U} \subseteq \mathcal{T}$ and hence, all definitions on $\mathcal{T}$ can be used on $\mathbb{U}$. We let $U, V, W, U_1, V_1, U', T, T_1, T_2, \ldots$ range over $\mathcal{T}$. We let $T, T_1, T_2, T', \ldots$ range over $\mathbb{T}$ and $U, V, W, U_1, V_1, U', \ldots$ range over $\mathbb{U}$.

   We quotient types by taking $\sqcap$ to be commutative (i.e. $U_1 \sqcap U_2 = U_2 \sqcap U_1$), associative (i.e. $U_1 \sqcap (U_2 \sqcap U_3) = (U_1 \sqcap U_2) \sqcap U_3$), idempotent (i.e. $U \sqcap U = U$) and the distributivity of expansion variables to $\sqcap$ (i.e. $e(U_1 \sqcap U_2) = eU_1 \sqcap eU_2$).

4. *We denote $e_{i_l} \ldots e_{i_n}$ by $\vec{e}_{i(l:n)}$ and $U_n \sqcap U_{n+1} \ldots \sqcap U_m$ by $\sqcap_{i=n}^{m} U_i$ $(n \leq m)$.*

5. *We define a function $d : \mathcal{T} \mapsto \mathbb{N}$ by (hence d is also defined on $\mathbb{U}$):*
   - $d(a) = 0$ • $d(U \to T) = \min(d(U), d(T))$
   - $d(eU) = d(U) + 1$ • $d(U \sqcap V) = \min(d(U), d(V))$.

   The function d is well defined because: $\forall\; n, m, k \in \mathbb{N}$,
   - $\min(n, m) = \min(m, n)$.
   - $\min(n, \min(m, k)) = \min(\min(n, m), k) = \min(n, m, k)$.
   - $\min(n, n) = n$.
   - $\min(n, m) + 1 = \min(n + 1, m + 1)$.

6. *We define the good types on $\mathcal{T}$ by (this also defines good types on $\mathbb{U}$):*
   - *If $a \in \mathcal{A}$, then a is good.*
   - *If U, T are good and $d(U) \geq d(T)$, then $U \to T$ is good.*
   - *If U, V are good and $d(U) = d(V)$, then $U \sqcap V$ is good.*
   - *If U is good and $e \in \mathcal{E}$, then eU is good.*

The next lemma states when arrow, intersection and expansion types are good.

Lemma 12 1. *On $\mathcal{T}$ (hence on $\mathbb{U}$), we have the following:*

   (a) *(U, T are good and $d(U) \geq d(T)$) iff $U \to T$ is good.*
   (b) *(U, V are good and $d(U) = d(V)$) iff $U \sqcap V$ is good.*
   (c) *U is good iff eU is good.*

2. *On $\mathbb{U}$, we have in addition the following:*

   (a) *If $T \in \mathbb{T}$, then $d(T) = 0$.*
   (b) *If $d(U) = n$ then $U = \sqcap_{i=1}^{k} \vec{e}_{i(1:n)} V_i$ where $k \geq 1$ and $\exists i. V_i \in \mathbb{T}$.*
   (c) *If U is good and $d(U) = n$, then $U = \sqcap_{i=1}^{k} \vec{e}_{i(1:n)} T_i$ where $k \geq 1$ and $\forall\; 1 \leq i \leq k$, $T_i \in \mathbb{T}$.*
   (d) *U and T are good iff $U \to T$ is good.*

Proof

1. The if direction is by definition. We only do the if direction.
   1a. By induction on the derivation of $U \to T$ good. 1b. By induction on the derivation of $U \sqcap V$ good. 1c. By induction on the derivation of eU good.

2. 2a. By induction on T. 2b. By induction on U. 2c. By induction on U. 2d. If) By 1. Only if) By 2, $\mathrm{d}(U) \geq 0 = \mathrm{d}(T)$. Hence, by 1, $U \to T$ is good.

$\square$

We now give the notion of an environment that will be used in our type systems.

Definition 13 (Environments) *1. A type environment is a set $\{x_i^{n_i} : U_i / 1 \leq i \leq n$ where $n \geq 0$ and $\forall 1 \leq i, j \leq n$, if $i \neq j$ then $x_i^{n_i} \neq x_j^{n_j}\}$. We denote such environment (call it Γ) by $x_1^{n_1} : U_1, x_2^{n_2} : U_2, \ldots, x_n^{n_n} : U_n$ or simply by $(x_i^{n_i} : U_i)_n$ and define $dom(\Gamma) = \{x_i^{n_i} / 1 \leq i \leq n\}$. We use $\Gamma, \Delta, \Gamma_1, \ldots$ to range over environments and write $()$ for the empty environment.*
Of course on $\mathcal{T}$, type environments take variables in $\mathcal{V}$ to $\mathcal{T}$. On $\mathbb{U}$, they take variables in $\mathcal{V}$ to $\mathbb{U}$. We say that:

- *Γ is good iff , for every $1 \leq i \leq k$, U_i is good.*
- *$d(\Gamma) > 0$ iff for every $1 \leq i \leq k$, $d(U_i) > 0$ and $n_i > 0$.*

2. *If $\Gamma = (x_i^{n_i} : U_i)_n$ and $x^m \notin dom(\Gamma)$, then we write $\Gamma, x^n : U$ for the type environment $x_1^{n_1} : U_1, \ldots, x_n^{n_n} : U_n, x^m : U$.*

3. *Let $\Gamma_1 = (x_i^{n_i} : U_i)_n, (y_j^{m_j} : V_j)_m$ and $\Gamma_2 = (x_i^{n_i} : U_i')_n, (z_k^{r_k} : W_k)_r$. We write $\Gamma_1 \sqcap \Gamma_2$ for the type environment $(x_i^{n_i} : U_i \sqcap U_i')_n, (y_j^{m_j} : V_j)_m, (z_k^{r_k} : W_k)_r$. Note that $dom(\Gamma_1 \sqcap \Gamma_2) = dom(\Gamma_1) \cup dom(\Gamma_2)$ and that $\sqcap$ is commutative, associative and idempotent on environments.*

4. *Let $\Gamma = (x_i^{n_i} : T_i)_n$. We let $e\Gamma = (x_i^{n_i+1} : eT_i)_n$. Note that $e(\Gamma_1 \sqcap \Gamma_2) = e\Gamma_1 \sqcap e\Gamma_2$.*

5. *We say that Γ_1 is joinable with Γ_2 and write $\Gamma_1 \diamond \Gamma_2$ iff*
   $\forall x \in \mathcal{V}$, if $x^m \in dom(\Gamma_1)$ and $x^n \in dom(\Gamma_2)$, then $m = n$.

3.2 The realisability semantics

Crucial to a realisability semantics is the notion of a saturated set defined below.

Definition 14 (Saturated sets) *Let $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{M}$.*

1. *We use $\mathcal{P}(\mathcal{X})$ to denote the powerset of $\mathcal{X}$, i.e. $\{\mathcal{Y} / \mathcal{Y} \subseteq \mathcal{X}\}$.*
2. *We let $\mathcal{X} \rightsquigarrow \mathcal{Y} = \{M \in \mathcal{M} / M\ N \in \mathcal{Y}$ for all $N \in \mathcal{X}$ such that $M \diamond N\}$.*
3. *We say that $\mathcal{X}$ is saturated iff whenever $M \rhd_\beta^* N$ and $N \in \mathcal{X}$, then $M \in \mathcal{X}$.*
4. *We say that $\mathcal{X} \wr \mathcal{Y}$ iff $\forall M \in \mathcal{X} \rightsquigarrow \mathcal{Y}, \exists N \in \mathcal{X}$ such that $M \diamond N$.*

The next lemma shows that saturation is closed under intersection, lifting and arrows. Moreover, the set of good terms of degree n is saturated for every n.

Lemma 15 *1. $(\mathcal{X} \cap \mathcal{Y})^+ = \mathcal{X}^+ \cap \mathcal{Y}^+$.*

2. *If $\mathcal{X}, \mathcal{Y}$ are saturated sets, then $\mathcal{X} \cap \mathcal{Y}$ is saturated.*
3. *If $\mathcal{X}$ is saturated, then $\mathcal{X}^+$ is saturated.*
4. *If $\mathcal{Y}$ is saturated, then, for every set $\mathcal{X}$, $\mathcal{X} \rightsquigarrow \mathcal{Y}$ is saturated.*
5. *(a) $(\mathcal{X} \rightsquigarrow \mathcal{Y})^+ \subseteq \mathcal{X}^+ \rightsquigarrow \mathcal{Y}^+$. (b)If $\mathcal{X}^+ \wr \mathcal{Y}^+$, then $\mathcal{X}^+ \rightsquigarrow \mathcal{Y}^+ \subseteq (\mathcal{X} \rightsquigarrow \mathcal{Y})^+$.*
6. *For every $n \in \mathbb{N}$, the set $\mathbb{M}^n$ is saturated.*

Proof 1. and 2. are easy.

3. If $M \rhd_\beta^* N^+$ where $N \in \mathcal{X}$, then, by lemma 8.6, $M = P^+$ and $P \rhd_\beta N$. As $\mathcal{X}$ is saturated, $P \in \mathcal{X}$ and so $P^+ = M \in \mathcal{X}^+$.

4. Let $M \in \mathcal{X} \rightsquigarrow \mathcal{Y}$ and $N \rhd_\beta^* M$. If $P \in \mathcal{X}$ such that $N \diamond P$, then $NP \rhd_\beta^* MP$ and, by lemma 8.8 $M \diamond P$. Since $(M\ P) \in \mathcal{Y}$ and $\mathcal{Y}$ is saturated, $(N\ P) \in \mathcal{Y}$. Hence, $N \in \mathcal{X} \rightsquigarrow \mathcal{Y}$.

5. (a) Let $M \in (\mathcal{X} \rightsquigarrow \mathcal{Y})^+$, then $M = N^+$ and $N \in \mathcal{X} \rightsquigarrow \mathcal{Y}$. If $P \in \mathcal{X}^+$ such that $M \diamond P$, then $P = Q^+$, $Q \in \mathcal{X}$ and $MP = N^+Q^+ = (NQ)^+$. By lemma 8.1(c)i, $N \diamond Q$ and hence $NQ \in \mathcal{Y}$ and $MP \in \mathcal{Y}^+$. Thus $M \in \mathcal{X}^+ \rightsquigarrow \mathcal{Y}^+$.

   (b) let $M \in \mathcal{X}^+ \rightsquigarrow \mathcal{Y}^+$. There is $N \in \mathcal{X}^+$ such that $M \diamond N$. We have $MN \in \mathcal{Y}^+$, then $MN = P^+$ where $P \in \mathcal{Y}$. Hence, $M = M_1^+$. Let $N_1 \in \mathcal{X}$ such that $M_1 \diamond N_1$. By lemma 8.1(c)i, $M \diamond N_1^+$ and we have $(M_1N_1)^+ = M_1^+N_1^+ \in \mathcal{Y}^+$. Hence $M_1N_1 \in \mathcal{Y}$. Thus $M_1 \in \mathcal{X} \rightsquigarrow \mathcal{Y}$ and $M = M_1^+ \in (\mathcal{X} \rightsquigarrow \mathcal{Y})^+$.

6. If $M \rhd_\beta^* N$ and $N \in \mathbb{M} \cap \mathcal{M}^n$ then by lemma 6.(1 and 2), $M \in \mathbb{M} \cap \mathcal{M}^n$. $\square$

Now we give the basic step in our realisability semantics: the interpretations and meanings of types.

Definition 16 (Interpretations and meaning of types) *Let $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ where $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ and $\mathcal{V}_1, \mathcal{V}_2$ are both denumerably infinite.*

1. *Let $x \in \mathcal{V}_1$ and $n \in \mathbb{N}$. We define $\mathcal{N}_x^n = \{x^n\ N_1...N_k \in \mathbb{M}\ /\ k \geq 0\}$.*
   It is easy to show that if $x^n\ N_1...N_k \in \mathcal{N}_x^n$ then $\forall\ 1 \leq i \leq k$, $d(N_i) \geq n$.

2. *An interpretation $\mathcal{I} : \mathcal{A} \mapsto \mathcal{P}(\mathcal{M}^0)$ is a function such that for all $a \in \mathcal{A}$:*
   • $\mathcal{I}(a)$ is saturated and • $\forall x \in \mathcal{V}_1$, $\mathcal{N}_x^0 \subseteq \mathcal{I}(a) \subseteq \mathbb{M}^0$.

3. *Let an interpretation $\mathcal{I} : \mathcal{A} \mapsto \mathcal{P}(\mathcal{M}^0)$ We extend an interpretation $\mathcal{I}$ to $\mathcal{T}$ (hence this includes $\mathbb{U}$) as follows: • $\mathcal{I}(eU) = \mathcal{I}(U)^+$*
   • $\mathcal{I}(U \sqcap V) = \mathcal{I}(U) \cap \mathcal{I}(V)$ • $\mathcal{I}(U \to T) = \mathcal{I}(U) \rightsquigarrow \mathcal{I}(T)$

   Since $\cap$ is commutative, associative and idempotent, then, by lemma 15.1, the function $\mathcal{I}$ is well defined.

4. *Let $U \in \mathcal{T}$ (hence U can be in $\mathbb{U}$). We define the meaning $[U]$ of U by:*

$$[U] = \{M \in \mathcal{M}\ /\ M \textit{ is closed and } M \in \bigcap_{\mathcal{I}\ \textit{interpretation}} \mathcal{I}(U)\}$$

The next lemma shows that type interpretations are saturated and interpretations of good types only contain good terms.

Lemma 17 *On $\mathcal{T}$ (hence also on $\mathbb{U}$) we have the following:*

1. *For any type U and interpretation $\mathcal{I}$, we have $\mathcal{I}(U)$ is saturated.*

2. *If U is a good type such that $\mathrm{d}(U) = n$ and $\mathcal{I}$ is an interpretation, then $\forall x \in \mathcal{V}_1$, $x^n \in \mathcal{N}_x^n \subseteq \mathcal{I}(U) \subseteq \mathbb{M}^n$.*

Proof 1. By induction on U using lemma 15.
2.Obviously, $x^n \in \mathcal{N}_x^n$. We prove $\mathcal{N}_x^n \subseteq \mathcal{I}(U) \subseteq \mathbb{M}^n$ by induction on U good. Case $U = a$: by definition. Case $U = U \sqcap V$ (resp. $U = eU'$): use IH since U, V are good and $\mathrm{d}(U) = \mathrm{d}(V)$ (resp. U' is good, $\mathrm{d}(U) = \mathrm{d}(U') + 1$, $(\mathcal{N}_x^n)^+ = \mathcal{N}_x^{n+1}$ and $(\mathcal{M}^n)^+ = \mathcal{M}^{n+1}$).
Case $U = U \to T$: by definition, U, T are good and $m = \mathrm{d}(U) \geq \mathrm{d}(T) = n$.

- Let $N_1, ..., N_k$ such that $x^n N_1...N_k \in \mathbb{M}$ (note that $\mathrm{d}(x^n N_1...N_k) = n$) and let $N \in \mathcal{I}(U)$ such that $(x^n N_1...N_k) \diamond N$ (hence $x^n N_1...N_k N \in \mathcal{M}$). By IH, $\mathrm{d}(N) = m \geq n$ and $N \in \mathbb{M}$. Hence, $x^n N_1...N_k N \in \mathbb{M}$ and $x^n N_1...N_k N \in \mathcal{N}_x^n$. By IH, $x^n N_1...N_k N \in \mathcal{I}(T)$. Thus $x^n N_1...N_k \in \mathcal{I}(U \to T)$.

- Let $M \in \mathcal{I}(U \to T)$. Let $x \in \mathcal{V}_1$ such that $\forall\ p \in \mathbb{N}$, $x^p \notin FV(M)$. Hence, $M \diamond x^m$. By IH, $x^m \in \mathcal{I}(U)$. Then $M\ x^m \in \mathcal{I}(T)$, and so by IH $M\ x^m \in \mathbb{M}^n$. By lemma 2, M is good and $\mathrm{d}(M) \leq m$. Since $\mathrm{d}(M\ x^m) = \min(\mathrm{d}(M), m) = n$, $\mathrm{d}(M) = n$ and so $M \in \mathbb{M}^n$. □

Corollary 18 (Meanings of good types consist of good terms) *On $\mathcal{T}$ (hence also on $\mathbb{U}$) we have: If U is a good type such that $\mathrm{d}(U) = n$ then $[U] \subseteq \mathbb{M}^n$.*

Proof Simply note that by lemma 17, for any interpretation $\mathcal{I}$, $\mathcal{I}(U) \subseteq \mathbb{M}^n$. □

Lemma 19 (The meaning of types is closed under type operations)
On $\mathcal{T}$ (hence also on $\mathbb{U}$) the following hold:

1. $[eU] = [U]^+$
2. $[U \sqcap V] = [U] \cap [V]$
3. *If $U \to T$ is good, then for any interpretation $\mathcal{I}$, $\mathcal{I}(U) \wr \mathcal{I}(T)$.*
4. *On $\mathcal{T}$ only (since $eU \to eT \notin \mathbb{U}$), we have:*
   If $U \to T$ is good, then $[e(U \to T)] = [eU \to eT]$.

Proof 1. and 2. are easy. 3. Let $\mathrm{d}(U) = n$ and $M \in \mathcal{I}(U) \rightsquigarrow \mathcal{I}(T)$. Take $x \in \mathcal{V}_1$ such that $\forall p \in \mathbb{N}$, $x^p \notin FV(M)$. Hence, $M \diamond x^n$. By lemma 12, U is good and by lemma 17, $x^n \in \mathcal{I}(U)$.

4. Since $U \to T$ is good, then, by lemma 12, U, T are good and $\mathrm{d}(U) \geq \mathrm{d}(T)$. Again by lemma 12, eU, eT are good, $\mathrm{d}(eU) \geq \mathrm{d}(eT)$ and $eU \to eT$ is good. Hence by 3. above, $\mathcal{I}(U)^+ \wr \mathcal{I}(T)^+$. Thus, by lemma 15.5, for any interpretation $\mathcal{I}$ we have $\mathcal{I}(e(U \to T)) = \mathcal{I}(eU \to eT)$. □

4 The typing systems $\vdash_1$ and $\vdash_2$

In this section we introduce $\vdash_1$ and $\vdash_2$, our two intersection type systems with expansion variables. In $\vdash_1$, types are not restricted and subject reduction fails. In $\vdash_2$, the syntax of types is restricted in the sense that arrows cannot occur to the left of intersections or expansions. In order to guarantee the subject reduction property for this type system (and hence completeness later on), we introduce a subtyping relation which will allow intersection type elimination (something not available in the first type system).

4.1 The typing rules

In this section we introduce the typing rules and establish a number of properties including the generation lemma and that when a term is typable then it, and its type and its context are all good and they all satisfy the relevant hierarchy between types and terms and no redexes are blocked.

Definition 20 *The type system $\vdash_1$ (resp. $\vdash_2$) uses the set $\mathcal{T}$ (resp. $\mathbb{U}$) of definition 11. We follow [3] and write type judgements as $M : \langle \Gamma \vdash U \rangle$ instead of the traditional format of $\Gamma \vdash M : U$. For $i \in \{1, 2\}$, the typing rules of $\vdash_i$ are (recall that when used for $\vdash_1$, U and T range over $\mathcal{T}$, and when used for $\vdash_2$, U ranges over $\mathbb{U}$ and T ranges over $\mathbb{T}$) given on the lefthand side of figure 4.1. In the last clause, the binary relation $\sqsubseteq$ is defined on U by the rules on the righthand side of figure 4.1. We let Φ denote types in $\mathbb{U}$, or environments Γ or typings $\langle \Gamma \vdash_2 U \rangle$. When $\Phi \sqsubseteq \Phi'$, then Φ and Φ' belong to the same set ($\mathbb{U}$/environments/typings).*

Let Γ be a type environment, $i \in \{1, 2\}$, $U \in \mathcal{T}$ and $M \in \mathcal{M}$. We say that:

$$\frac{T \text{ good} \quad \mathrm{d}(T)=n}{x^n : \langle (x^n : T) \vdash_1 T \rangle}\ (ax)$$ $$\frac{T \text{ good}}{x^0 : \langle (x^0 : T) \vdash_2 T \rangle}\ (ax)$$ $$\frac{M : \langle \Gamma, (x^n : U) \vdash_i T \rangle}{\lambda x^n.M : \langle \Gamma \vdash_i U \to T \rangle}\ (\to_I)$$ $$\frac{M_1 : \langle \Gamma_1 \vdash_i U \to T \rangle \quad M_2 : \langle \Gamma_2 \vdash_i U \rangle \quad \Gamma_1 \diamond \Gamma_2}{M_1 M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_i T \rangle}\ (\to_E)$$ $$\frac{M : \langle \Gamma_1 \vdash_i U_1 \rangle \quad M : \langle \Gamma_2 \vdash_i U_2 \rangle}{M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_i U_1 \sqcap U_2 \rangle}\ (\sqcap)$$ $$\frac{M : \langle \Gamma \vdash_i U \rangle}{M^+ : \langle e\Gamma \vdash_i eU \rangle}\ (exp)$$ $$\frac{M : \langle \Gamma \vdash_2 U \rangle \quad \langle \Gamma \vdash_2 U \rangle \sqsubseteq \langle \Gamma' \vdash_2 U' \rangle}{M : \langle \Gamma' \vdash_2 U' \rangle}\ (\sqsubseteq)$$	$$\frac{}{\Phi \sqsubseteq \Phi}\ (ref)$$ $$\frac{\Phi_1 \sqsubseteq \Phi_2 \quad \Phi_2 \sqsubseteq \Phi_3}{\Phi_1 \sqsubseteq \Phi_3}\ (tr)$$ $$\frac{U_2 \text{ good} \quad \mathrm{d}(U_1)=\mathrm{d}(U_2)}{U_1 \sqcap U_2 \sqsubseteq U_1}\ (\sqcap_e)$$ $$\frac{U_1 \sqsubseteq V_1 \quad U_2 \sqsubseteq V_2}{U_1 \sqcap U_2 \sqsubseteq V_1 \sqcap V_2}\ (\sqcap)$$ $$\frac{U_2 \sqsubseteq U_1 \quad T_1 \sqsubseteq T_2}{U_1 \to T_1 \sqsubseteq U_2 \to T_2}\ (\to)$$ $$\frac{U_1 \sqsubseteq U_2}{eU_1 \sqsubseteq eU_2}\ (\sqsubseteq_{exp})$$ $$\frac{U_1 \sqsubseteq U_2}{\Gamma, (y^n : U_1) \sqsubseteq \Gamma, (y^n : U_2)}\ (\sqsubseteq_c)$$ $$\frac{U_1 \sqsubseteq U_2 \quad \Gamma_2 \sqsubseteq \Gamma_1}{\langle \Gamma_1 \vdash_2 U_1 \rangle \sqsubseteq \langle \Gamma_2 \vdash_2 U_2 \rangle}\ (\sqsubseteq_{\langle\rangle})$$

Figure 1: Typing rules / Subtyping rules

- *Γ is $\vdash_i$-legal iff there are M, U such that $M : \langle \Gamma \vdash_i U \rangle$.*
- *$\langle \Gamma \vdash_i U \rangle$ is good iff Γ and U are good.*
- *$d(\langle \Gamma \vdash_i U \rangle) > 0$ iff $d(\Gamma) > 0$ and $d(U) > 0$.*

The next lemma establishes needed properties of the relation $\sqsubseteq$ on $\mathbb{U}$.

Lemma 21 *1. If $\Gamma \sqsubseteq \Gamma'$, $U \sqsubseteq U'$ and $x^n \notin dom(\Gamma)$ then $\Gamma, (x^n : U) \sqsubseteq \Gamma', (x^n : U')$.*

2. $\Gamma \sqsubseteq \Gamma'$ iff $\Gamma = (x_i^{n_i} : U_i)_n$, $\Gamma' = (x_i^{n_i} : U_i')_n$ and for every $1 \le i \le n$, $U_i \sqsubseteq U_i'$.

3. $\langle \Gamma \vdash_2 U \rangle \sqsubseteq \langle \Gamma' \vdash_2 U' \rangle$ iff $\Gamma' \sqsubseteq \Gamma$ and $U \sqsubseteq U'$.

4. Let $U_1 \sqsubseteq U_2$.

(a) $d(U_1) = d(U_2)$.

(b) U_1 is good iff U_2 is good.

(c) If U_2 is good and $d(U_2) = n$, then $U_1 = \sqcap_{i=1}^k \vec{e}_{i(1:n)} T_i$, $U_2 = \sqcap_{j=1}^p \vec{e'}_{j(1:n)} T_j'$, where $p, k \ge 1$, $\forall 1 \le i \le k$ $T_i \in \mathbb{T}$, $\forall 1 \le j \le p$ $T_j' \in \mathbb{T}$ and $\forall 1 \le j \le p$, $\exists 1 \le i \le k$ such that $\vec{e}_{i(1:n)} = \vec{e'}_{j(1:n)}$ and $T_i \sqsubseteq T_j'$.

(d) Let $U_1 = \sqcap_{i=1}^k \vec{e}_{i(1:n_i)} (V_i \to T_i)$ and $U_2 = \sqcap_{j=1}^p \vec{e'}_{j(1:m_j)} (V_j' \to T_j')$. If U_1 is good and $d(U_1) = n$ then $\forall i, j$, $n_i = m_j = n$ and $\forall 1 \le j \le p$, $\exists 1 \le i \le k$ such that $\vec{e}_{i(1:n)} = \vec{e'}_{j(1:n)}$, $V_j' \sqsubseteq V_i$ and $T_i \sqsubseteq T_j'$.

5. If $U \sqsubseteq V \sqcap a$, then $U = U' \sqcap a$.

6. If $eU \sqsubseteq V$ then $V = eU'$ where $U \sqsubseteq U'$.

7. If $U \to T \sqsubseteq V$ and $U \to T$ is good, then $V = \sqcap_{i=1}^p (U_i \to T_i)$ where $p \ge 1$ and $\forall 1 \le i \le p$, $U_i \sqsubseteq U$ and $T \sqsubseteq T_i$.

8. If $\sqcap_{i=1}^k \vec{e}_{i(1:n_i)} (V_i \to T_i) \sqsubseteq V$ where V is good, $d(V) = n$ and $k \ge 1$ then $\forall i, n_i = n$ and $V = \sqcap_{i=1}^p \vec{e'}_{i(1:n)} (V_i' \to T_i')$ where $p \ge 1$ and $\forall 1 \le i \le p$, $\exists 1 \le j \le k$ such that $\vec{e}_{j(1:n)} = \vec{e'}_{i(1:n)}$, $V_i' \sqsubseteq V_j$ and $T_j \sqsubseteq T_i'$.

9. *Let* $\Phi_1 \sqsubseteq \Phi_2$.
   - $d(\Phi_1) > 0$ *iff* $d(\Phi_2) > 0$
   - Φ_1 *is good iff* Φ_2 *is good.*

10. *If* $U \sqsubseteq U'_1 \sqcap U'_2$ *then* $U = U_1 \sqcap U_2$ *where* $U_1 \sqsubseteq U'_1$ *and* $U_2 \sqsubseteq U'_2$.

11. *If* $\Gamma \sqsubseteq \Gamma'_1 \sqcap \Gamma'_2$ *then* $\Gamma = \Gamma_1 \sqcap \Gamma_2$ *where* $\Gamma_1 \sqsubseteq \Gamma'_1$ *and* $\Gamma_2 \sqsubseteq \Gamma'_2$.

Proof

1. Show by induction on the derivation of $\Gamma \sqsubseteq \Gamma'$ that if $\Gamma \sqsubseteq \Gamma'$ and $\Gamma, (x^n : U)$ is an environment, then $\Gamma, (x^n : U) \sqsubseteq \Gamma', (x^n : U)$. Then use tr.

2. Only if) By induction on the derivation of $\Gamma \sqsubseteq \Gamma'$. If) By induction on n and 1.

3. Only if) By induction on the derivation of $\langle \Gamma \vdash_2 U \rangle \sqsubseteq \langle \Gamma' \vdash_2 U' \rangle$. If) By $\sqsubseteq_{\langle\rangle}$.

4. By induction on the derivation of $U_1 \sqsubseteq U_2$ using lemmas 12.2 and 12. We do case tr of 4d. If $\dfrac{\sqcap_{i=1}^k \vec{e}_{i(1:n_i)}(V_i \to T_i) \sqsubseteq V \quad V \sqsubseteq \sqcap_{j=1}^p \vec{e'}_{j(1:m_j)}(V'_j \to T'_j)}{\sqcap_{i=1}^k \vec{e}_{i(1:n_i)}(V_i \to T_i) \sqsubseteq \sqcap_{j=1}^p \vec{e'}_{j(1:m_j)}(V'_j \to T'_j)}$, then, by 4c, $\forall i, n_i = n$ and $V = \sqcap_{l=1}^q \vec{e''}_{l(1:n)} T''_l$ where $q \geq 1$ and $\forall 1 \leq l \leq q$, $\exists 1 \leq i \leq k$, such that $\vec{e''}_{l(1:n)} = \vec{e}_{i(1:n)}$ and $V_i \to T_i \sqsubseteq T''_l$. If $T''_l = a$, then, by 5, $V_i \to T_i = V' \sqcap a$. Absurd. Hence, $\forall 1 \leq l \leq q$, $T''_l = W_l \to T'''_l$ and $V = \sqcap_{l=1}^q \vec{e''}_{l(1:n)}(W_l \to T'''_l)$. By IH, $\forall 1 \leq l \leq q$, $\exists 1 \leq i \leq k$ such that $\vec{e}_{i(1:n)} = \vec{e''}_{l(1:n)}$, $W_l \sqsubseteq V_i$ and $T_i \sqsubseteq T'''_l$. Also, by IH, $\forall j, m_j = m$ and $\forall 1 \leq j \leq p$, $\exists 1 \leq l \leq q$, $\vec{e''}_{l(1:n)} = \vec{e'}_{j(1:n)}$, $V'_j \sqsubseteq W_l$ and $T'''_l \sqsubseteq T'_j$. Hence, $\forall 1 \leq j \leq p$, $\exists 1 \leq i \leq k$, such that $\vec{e'}_{j(1:n)} = \vec{e}_{i(1:n)}$, $V'_j \sqsubseteq V_i$ and $T_i \sqsubseteq T'_j$.

5. By induction on $U \sqsubseteq V \sqcap a$.

6. By induction on $eU \sqsubseteq V$.

7. By 4c, $V = \sqcap_{i=1}^p T'_i$ where $p \geq 1$ and $\forall 1 \leq i \leq p$, $U \to T \sqsubseteq T'_i$. If $T'_i = a$, then, by 5, $U \to T = U' \sqcap a$. Absurd. Hence, $T'_i = U_i \to T_i$. Hence, by 4d, $\forall 1 \leq i \leq p$, $U_i \sqsubseteq U$ and $T \sqsubseteq T_i$.

8. By 4c, $\forall i, n_i = n$ and $V = \sqcap_{i=1}^p \vec{e'}_{i(1:n)} T''_i$ where $p \geq 1$ and $\forall 1 \leq i \leq p$, $\exists 1 \leq j_i \leq k$ such that $\vec{e}_{j_i(1:n)} = \vec{e'}_{i(1:n)}$ and $V_{j_i} \to T_{j_i} \sqsubseteq T''_i$. Let $1 \leq i \leq p$. If $T''_i = a$, then, by 5, $V_{j_i} \to T_{j_i} = U' \sqcap a$. Absurd. Hence, $T''_i = V'_i \to T'_i$. By 7, $V'_i \sqsubseteq V_{j_i}$ and $T_{j_i} \sqsubseteq T'_i$. We are done.

9. Using 4. and lemma 21.

10. By induction on $U \sqsubseteq U'_1 \sqcap U'_2$.
   - Let $\dfrac{}{U'_1 \sqcap U'_2 \sqsubseteq U'_1 \sqcap U'_2}$. By ref, $U'_1 \sqsubseteq U'_1$ and $U'_2 \sqsubseteq U'_2$.
   - Let $\dfrac{U \sqsubseteq U'' \qquad U'' \sqsubseteq U'_1 \sqcap U'_2}{U \sqsubseteq U'_1 \sqcap U'_2}$. By IH, $U'' = U''_1 \sqcap U''_2$ such that $U''_1 \sqsubseteq U'_1$ and $U''_2 \sqsubseteq U'_2$. Again by IH, $U = U_1 \sqcap U_2$ such that $U_1 \sqsubseteq U''_1$ and $U_2 \sqsubseteq U''_2$. So by tr, $U_1 \sqsubseteq U'_1$ and $U_2 \sqsubseteq U'_2$.
   - Let $\dfrac{U \textit{ good } \ \& \ \mathrm{d}(U'_1 \sqcap U'_2) = \mathrm{d}(U)}{(U'_1 \sqcap U'_2) \sqcap U \sqsubseteq U'_1 \sqcap U'_2}$. By ref, $U'_1 \sqsubseteq U'_1$ and $U'_2 \sqsubseteq U'_2$. Moreover:

* If $\mathrm{d}(U) = \mathrm{d}(U_1' \sqcap U_2') = \mathrm{d}(U_1')$ then by $\sqcap_e$, $U_1' \sqcap U \sqsubseteq U_1'$. We are done.
* If $\mathrm{d}(U) = \mathrm{d}(U_1' \sqcap U_2') = \mathrm{d}(U_2')$ then by $\sqcap_e$, $U_2' \sqcap U \sqsubseteq U_2'$. We are done.

– If $\dfrac{U_1 \sqsubseteq U_1' \quad \& \quad U_2 \sqsubseteq U_2'}{U_1 \sqcap U_2 \sqsubseteq U_1' \sqcap U_2'}$ there is nothing to prove.

– If $\dfrac{U \sqsubseteq U_1' \sqcap U_2'}{eU \sqsubseteq eU_1' \sqcap eU_2'}$ then by IH $U = U_1 \sqcap U_2$ such that $U_1 \sqsubseteq U_1'$ and $U_2 \sqsubseteq U_2'$. So, $eU = eU_1 \sqcap eU_2$ and by $\sqsubseteq_{exp}$, $eU_1 \sqsubseteq eU_1'$ and $eU_2 \sqsubseteq eU_2'$.

11. By induction on $\Gamma \sqsubseteq \Gamma_1' \sqcap \Gamma_2'$.

– Let $\dfrac{}{\Gamma_1' \sqcap \Gamma_2' \sqsubseteq \Gamma_1' \sqcap \Gamma_2'}$. By ref, $\Gamma_1' \sqsubseteq \Gamma_1'$ and $\Gamma_2' \sqsubseteq \Gamma_2'$.

– Let $\dfrac{\Gamma \sqsubseteq \Gamma'' \qquad \Gamma'' \sqsubseteq \Gamma_1' \sqcap \Gamma_2'}{\Gamma \sqsubseteq \Gamma_1' \sqcap \Gamma_2'}$. By IH, $\Gamma'' = \Gamma_1'' \sqcap \Gamma_2''$ such that $\Gamma_1'' \sqsubseteq \Gamma_1'$ and $\Gamma_2'' \sqsubseteq \Gamma_2'$. Again by IH, $\Gamma = \Gamma_1 \sqcap \Gamma_2$ such that $\Gamma_1 \sqsubseteq \Gamma_1''$ and $\Gamma_2 \sqsubseteq \Gamma_2''$. So by tr, $\Gamma_1 \sqsubseteq \Gamma_1'$ and $\Gamma_2 \sqsubseteq \Gamma_2'$.

– Let $\dfrac{U_1 \sqsubseteq U_2}{\Gamma, (y^n : U_1) \sqsubseteq \Gamma, (y^n : U_2)}$ where $\Gamma, (y^n : U_2) = \Gamma_1' \sqcap \Gamma_2'$.

* If $\Gamma_1' = \Gamma_1'', (y^n : U_2')$ and $\Gamma_2' = \Gamma_2'', (y^n : U_2'')$ such that $U_2 = U_2' \sqcap U_2''$, then by 10, $U_1 = U_1' \sqcap U_1''$ such that $U_1' \sqsubseteq U_2'$ and $U_1'' \sqsubseteq U_2''$. Hence $\Gamma = \Gamma_1'' \sqcap \Gamma_2''$ and $\Gamma, (y^n : U_1) = \Gamma_1 \sqcap \Gamma_2$ where $\Gamma_1 = \Gamma_1'', (y^n : U_1')$ and $\Gamma_2 = \Gamma_2'', (y^n : U_1'')$ such that $\Gamma_1 \sqsubseteq \Gamma_1'$ and $\Gamma_2 \sqsubseteq \Gamma_2'$ by $\sqsubseteq_c$.
* If $y^n \notin dom(\Gamma_1')$ then $\Gamma = \Gamma_1' \sqcap \Gamma_2''$ where $\Gamma_2'', (y^n : U_2) = \Gamma_2'$. Hence, $\Gamma, (y^n : U_1) = \Gamma_1' \sqcap \Gamma_2$ where $\Gamma_2 = \Gamma_2'', (y^n : U_1)$. By ref and $\sqsubseteq_c$, $\Gamma_1' \sqsubseteq \Gamma_1'$ and $\Gamma_2 \sqsubseteq \Gamma_2'$.
* If $y^n \notin dom(\Gamma_2')$ then similar to the above case.

□

The next lemma is both a context lemma and a typability of subterms lemma.

Lemma 22 *Let $i \in \{1,2\}$ and $M : \langle \Gamma \vdash_i U \rangle$. We have:*

1. *(a) $FV(M) = dom(\Gamma)$.*
   (b) If $M : \langle \Delta \vdash_i V \rangle$, then $dom(\Gamma) = dom(\Delta)$.
2. *If $x^n : U_1 \in \Gamma$ and $y^m : U_2 \in \Gamma$, then:*
   (a) If $x^n : U_1 \neq y^m : U_2$, then $x^n \neq y^m$.
   (b) If $x = y$, then $n = m$ and $U_1 = U_2$.
3. *If $x^n : U_1 \in \Gamma$ and $y^m : U_2 \in \Gamma$ and $x^n : U_1 \neq y^m : U_2$, then $x \neq y$ and $x^n \neq y^m$.*
4. *Assume $N : \langle \Delta \vdash_i V \rangle$. We have $\Gamma \diamond \Delta$ iff $M \diamond N$.*
5. *If N is a subterm of M, then there are Δ, V such that $N : \langle \Delta \vdash_i V \rangle$.*
6. *If $\Gamma = \Gamma_1 \sqcap \Gamma_2 \sqcap \Gamma_3$, then $\Gamma_1 \diamond \Gamma_2$.*

Proof 1. (a) by induction on the derivation of $M : \langle \Gamma \vdash_i U \rangle$, (b) is a corollary of (a). 2. (a) and (b) by induction on the derivation of $M : \langle \Gamma \vdash_i U \rangle$ using 1. in the $\sqcap_i$ case. 3. is a corollary of 2. 4. use 1. 5. by induction on the derivation of $M : \langle \Gamma \vdash_i U \rangle$. 6. Let $\Gamma = (x_i^{n_i} : U_i)_n$ and $x \in \mathcal{V}$. If $x^p \in dom(\Gamma_1) \subseteq dom(\Gamma)$ and $x^q \in dom(\Gamma_2) \subseteq dom(\Gamma)$, then by 2, p=q. □

The next lemma shows that all typable terms are good, have good types, and have the same degree as their types. Moreover, all legal contexts are good.

Lemma 23 *Let $i \in \{1,2\}$. If $M : \langle (x_i^{n_i} : U_i)_n \vdash_i U \rangle$, then*

1. *$\forall\ 1 \le i \le n$, U_i is good and $d(U_i) = n_i \ge d(M)$.*
2. *U is good and $d(M) = d(U)$.*
3. *M is good.*

Proof By induction on the derivation of $M : \langle (x_i^{n_i} : U_i)_n \vdash_i U \rangle$.

- ax: 1. and 2. hold by the hypothesis of this rule. 3. holds by definition.
- Let $\dfrac{M : \langle \Gamma, (x^m : U) \vdash_i T \rangle}{\lambda x^m.M : \langle \Gamma \vdash_i U \to T \rangle}$ where $\Gamma = (x_i^{n_i} : U_i)_n$. By IH, $\mathrm{d}(U_i) = n_i$, $\mathrm{d}(U) = m$, $\mathrm{d}(M) = \mathrm{d}(T)$, $n_i, m \ge \mathrm{d}(M)$ and U_i, U, T are good. Hence $\mathrm{d}(U) \ge \mathrm{d}(T)$ and, by definition, $U \to T$ is good. Moreover, $\mathrm{d}(\lambda x^m.M) = \mathrm{d}(M) = \mathrm{d}(T) = \min(\mathrm{d}(U), \mathrm{d}(T)) = \mathrm{d}(U \to T)$ and $n_i \ge \mathrm{d}(\lambda x^m.M)$. Since by lemma 22, $FV(M) = dom(\Gamma, (x^m : U))$, $x^m \in FV(M)$ and since by IH, M is good, by definition $\lambda x^m.M$ is good.
- Let $\dfrac{M_1 : \langle \Gamma_1 \vdash_i U \to T \rangle \quad M_2 : \langle \Gamma_2 \vdash_i U \rangle \quad \Gamma_1 \diamond \Gamma_2}{M_1\ M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_i T \rangle}$ where $\Gamma_1 = (x_i^{n_i} : U_i)_n, (y_j^{m_j} : V_j)_m$, $\Gamma_2 = (x_i^{n_i} : U_i')_n, (z_k^{r_k} : W_k)_r$ and $\Gamma_1 \sqcap \Gamma_2 = (x_i^{n_i} : U_i \sqcap U_i')_n, (y_j^{m_j} : V_j)_m, (z_k^{r_k} : W_k)_r$. By IH, $\mathrm{d}(U_i) = \mathrm{d}(U_i') = n_i$, $\mathrm{d}(V_j) = m_j$, $\mathrm{d}(W_k) = r_k$, $\mathrm{d}(M_1) = \mathrm{d}(U \to T)$, $\mathrm{d}(M_2) = \mathrm{d}(U)$, $n_i, m_j \ge \mathrm{d}(M_1)$, $n_i, r_k \ge \mathrm{d}(M_2)$ and $U_i, V_j, U_i', W_k, U \to T, U$ are good. By definition, $U_i \sqcap U_i'$ and T are good. Also, $\mathrm{d}(U_i \sqcap U_i') = n_i$. Since $U \to T$ is good, then, by lemma 12, $\mathrm{d}(U) \ge \mathrm{d}(T)$, $\mathrm{d}(M_1) = \mathrm{d}(T)$ and $\mathrm{d}(M_1\ M_2) = \min(\mathrm{d}(M_1), \mathrm{d}(M_2)) = \min(\mathrm{d}(T), \mathrm{d}(U)) = \mathrm{d}(T)$. We have $n_i \ge \mathrm{d}(M_1) = \mathrm{d}(T) = \mathrm{d}(M_1\ M_2)$, $m_j \ge \mathrm{d}(M_1) = \mathrm{d}(M_1\ M_2)$ and $r_k \ge \mathrm{d}(M_2) = \mathrm{d}(U) \ge \mathrm{d}(T) = \mathrm{d}(M_1\ M_2)$. Finally, $\mathrm{d}(M_1) = deg(U \to T) = \mathrm{d}(T) \le \mathrm{d}(U) = \mathrm{d}(M_2)$, by lemma 22, $M_1 M_2 \in \mathcal{M}$ so $M_1 \diamond M_2$ and by IH, M_1 and M_2 are good, so by definition $M_1 M_2$ is good.
- Let $\dfrac{M : \langle (x_i^{n_i} : U_i)_n \vdash_i U \rangle \quad M : \langle (x_i^{n_i} : V_i)_n \vdash_i V \rangle}{M : \langle (x_i^{n_i} : U_i \sqcap V_i)_n \vdash_i U \sqcap V \rangle}$ (note lemma 22.1). By IH, $\mathrm{d}(U_i) = \mathrm{d}(V_i) = n_i$, $\mathrm{d}(M) = \mathrm{d}(U) = \mathrm{d}(V)$, $n_i \ge \mathrm{d}(M)$ and U_i, V_i, U, V are good. Hence, $\mathrm{d}(U_i \sqcap V_i) = n_i$ and $\mathrm{d}(M) = \mathrm{d}(U \sqcap V)$. Moreover, by definition, $U_i \sqcap V_i$ and $U \sqcap V$ are good. Finally, by IH, M is good.
- Let $\dfrac{M : \langle (x_i^{n_i} : U_i)_n \vdash_i U \rangle}{M^+ : \langle (x_i^{n_i+1} : eU_i)_n \vdash_i eU \rangle}$. By IH, $\mathrm{d}(U_i) = n_i$, $\mathrm{d}(M) = \mathrm{d}(U)$, $n_i \ge \mathrm{d}(M)$ and U_i, U are good. Hence, $\mathrm{d}(eU_i) = n_i + 1$, $\mathrm{d}(M^+) = \mathrm{d}(eU)$ and $n_i + 1 \ge \mathrm{d}(M^+)$. Moreover, by definition, eU_i and eU are good. Finally, By IH, M is good, so by lemma 8.1d, M^+ is good.
- Let $\dfrac{M : \langle \Gamma' \vdash_2 U' \rangle \quad \langle \Gamma' \vdash_2 U' \rangle \sqsubseteq \langle (x_i^{n_i} : U_i)_n \vdash_2 U \rangle}{M : \langle (x_i^{n_i} : U_i)_n \vdash_2 U \rangle}$. By lemma 21, $\Gamma' = (x_i^{n_i} : U_i')_n$, for every $1 \le i \le n$, $U_i \sqsubseteq U_i'$ and $U' \sqsubseteq U$. By IH, U' is good, $\mathrm{d}(M) = \mathrm{d}(U')$ and $\forall\ 1 \le i \le n$, $\mathrm{d}(U_i') = n_i$, $n_i \ge \mathrm{d}(M)$ and U_i' are good. By lemma 21, U is good, $\mathrm{d}(M) = \mathrm{d}(U)$ and $\forall\ 1 \le i \le n$, $\mathrm{d}(U_i) = n_i$ and U_i are good. Moreover, M is good by IH.

$\square$

Remark 24 *The rules $\sqcap'_i$ and ax' given below are derivable in $\vdash_2$:*

$$\frac{M : \langle \Gamma_1 \vdash_2 U_1 \rangle \qquad M : \langle \Gamma_2 \vdash_2 U_2 \rangle}{M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_2 U_1 \sqcap U_2 \rangle}\ \sqcap'_i$$

$$\frac{U \textit{ is good} \qquad d(U) = n}{x^n : \langle (x^n : U) \vdash_2 U \rangle}\ ax'$$

Proof $\sqcap'_i$. Let $M : \langle \Gamma_1 \vdash_2 U_1 \rangle$ and $M : \langle \Gamma_2 \vdash_2 U_2 \rangle$. By lemma 22, $dom(\Gamma_1) = dom(\Gamma_2)$. Let $\Gamma_1 = (x_i^{n_i}, V_i)_n$ and $\Gamma_2 = (x_i^{n_i}, V'_i)_n$. By lemma 23, $\forall 1 \leq i \leq n$, V_i and V'_i are good and $\mathrm{d}(V_i) = \mathrm{d}(V'_i) = n_i$. By $\sqcap_e$, $V_i \sqcap V'_i \sqsubseteq V_i$ and $V_i \sqcap V'_i \sqsubseteq V'_i$. Hence, by lemma 21.2, $\Gamma_1 \sqcap \Gamma_2 \sqsubseteq \Gamma_1$ and $\Gamma_1 \sqcap \Gamma_2 \sqsubseteq \Gamma_2$ and by $\sqsubseteq$ and $\sqsubseteq_{\langle\rangle}$, $M : \langle \Gamma_1 \sqcap \Gamma_2, U_1 \rangle$ and $M : \langle \Gamma_1 \sqcap \Gamma_2, U_2 \rangle$. Finally, by $\sqcap_i$, $M : \langle \Gamma_1 \sqcap \Gamma_2, U_1 \sqcap U_2 \rangle$.
ax'. By lemma 12.2, $U = \sqcap_{i=1}^{k} \vec{e}_{i(1:n)} T_i$ where $k \geq 1$, and $\forall 1 \leq i \leq k$, $T_i \in \mathbb{T}$ and T_i is good. Let $1 \leq i \leq k$. By lemma 12.2, $\mathrm{d}(T_i) = 0$ and by ax, $x^0 : \langle (x^0 : T_i) \vdash_2 T_i \rangle$. Hence, $x^n : \langle (x^n : \vec{e}_{i(1:n)} T_i) \vdash_2 \vec{e}_{i(1:n)} T_i \rangle$ by n applications of exp. Now, by $k-1$ applications of $\sqcap'_i$, $x^n : \langle (x^n : U) \vdash_2 U \rangle$. □

Next, the generation lemma which says how type derivations are generated (recall that in $\mathcal{T}$, T ranges over all of $\mathcal{T}$ whereas in $\mathbb{U}$, T ranges only over $\mathbb{T}$).

Lemma 25 (Generation for $\vdash_1$)

1. *If $x^n : \langle \Gamma \vdash_1 T \rangle$, then $\Gamma = (x^n : T)$.*
2. *If $\lambda x^n.M : \langle \Gamma \vdash_1 T_1 \to T_2 \rangle$, then $M : \langle \Gamma, x^n : T_1 \vdash_1 T_2 \rangle$.*
3. *If $MN : \langle \Gamma \vdash_1 T \rangle$ then $\Gamma = \Gamma_1 \sqcap \Gamma_2$, $T = \sqcap_{i=1}^{n} \vec{e}_{i(1:m_i)} T_i$, $n \geq 1, m_i \geq 0$, $M : \langle \Gamma_1 \vdash_1 \sqcap_{i=1}^{n} \vec{e}_{i(1:m_i)} (T'_i \to T_i) \rangle$ and $N : \langle \Gamma_2 \vdash_1 \sqcap_{i=1}^{n} \vec{e}_{i(1:m_i)} T'_i \rangle$.*

Proof

1. By induction on the derivation of $x^n : \langle \Gamma \vdash_1 T \rangle$.
2. First, we prove by induction on the derivation of $\lambda x^n.M : \langle \Gamma \vdash_1 T_1 \to T_2 \rangle$ that $\exists k \geq 1, \Gamma_1, \Gamma_2, \ldots, \Gamma_k$, such that $\Gamma = \Gamma_1 \sqcap \Gamma_2 \ldots \sqcap \Gamma_k$ and $\forall\ 1 \leq i \leq k$, $M : \langle \Gamma_i, x^n : T_1 \vdash_1 T_2 \rangle$. We have two cases:
   - Case $\to_i$: take $k = 1$.
   - Case $\sqcap_i$: Let $\dfrac{\lambda x^n.M : \langle \Delta \vdash_1 T_1 \to T_2 \rangle \quad \lambda x^n.M : \langle \Omega \vdash_1 T_1 \to T_2 \rangle}{\lambda x^n.M : \langle \Delta \sqcap \Omega \vdash_1 T_1 \to T_2 \rangle}$. By IH, $\Delta = \Delta_1 \ldots \Delta_{k_1}$ and $\forall\ 1 \leq i \leq k_1$, $M : \langle \Delta_i, x^n : T_1 \vdash_1 T_2 \rangle$ and $\Omega = \Omega_1 \ldots \Omega_{k_2}$ and $\forall\ 1 \leq j \leq k_2$, $M : \langle \Omega_j, x^n : T_1 \vdash_1 T_2 \rangle$ and we are done.

   Now we prove 2. Since $\Gamma = \Gamma_1 \sqcap \Gamma_2 \ldots \sqcap \Gamma_k$ where $\forall\ 1 \leq i \leq k$, $M : \langle \Gamma_i, x^n : T_1 \vdash T_2 \rangle$, by $k-1$ applications of $\sqcap_i$ we get $M : \langle \Gamma, x^n : T_1 \vdash_1 T_2 \rangle$.
3. By induction on the derivation of $MN : \langle \Gamma \vdash_1 T \rangle$.

□

Lemma 26 (Generation for $\vdash_2$)

1. *If $x^n : \langle \Gamma \vdash_2 U \rangle$, then $\Gamma = (x^n : V)$ where $V \sqsubseteq U$.*
2. *If $\lambda x^n.M : \langle \Gamma \vdash_2 U \rangle$ and $d(U) = m$, then $U = \sqcap_{i=1}^{k} \vec{e}_{i(1:m)} (V_i \to T_i)$ where $k \geq 1$ and $\forall 1 \leq i \leq k$, $M : \langle \Gamma, x^n : \vec{e}_{i(1:m)} V_i \vdash_2 \vec{e}_{i(1:m)} T_i \rangle$.*

Proof 1. By induction on the derivation of $x^n : \langle \Gamma \vdash_2 U \rangle$.
2. By induction on the derivation of $\lambda x^n.M : \langle \Gamma \vdash_2 U \rangle$. We have four cases:

- If $\dfrac{M : \langle \Gamma, x^n : U \vdash_2 T \rangle}{\lambda x^n.M : \langle \Gamma \vdash_2 U \to T \rangle}$, nothing to prove.

- Let $\dfrac{\lambda x^n.M : \langle \Gamma \vdash_2 U_1\rangle \quad \lambda x^n.M : \langle \Gamma \vdash_2 U_2\rangle}{\lambda x^n.M : \langle \Gamma \vdash_2 U_1 \sqcap U_2\rangle}$. By lemma 23, $U_1 \sqcap U_2$ is good and $\mathrm{d}(U_1) = \mathrm{d}(U_2) = m$. By IH we have: $U_1 = \sqcap_{i=1}^k \vec{e}_{i(1:m)}(V_i \to T_i)$, $U_2 = \sqcap_{i=k+1}^{k+l} \vec{e}_{i(1:m)}(V_i \to T_i)$ (hence $U_1 \sqcap U_2 = \sqcap_{i=1}^{k+l} \vec{e}_{i(1:m)}(V_i \to T_i)$) where $k, l \geq 1$ and $\forall 1 \leq i \leq k+l$, $M : \langle \Gamma, x^n : \vec{e}_{i(1:m)} V_i \vdash_2 \vec{e}_{i(1:m)} T_i\rangle$. We are done.

- Let $\dfrac{\lambda x^n.M : \langle \Gamma \vdash_2 U\rangle}{\lambda x^{n+1}.M^+ : \langle e\Gamma \vdash_2 eU\rangle}$. By IH, $U = \sqcap_{i=1}^k \vec{e}_{i(1:m-1)}(V_i \to T_i)$ (since $\mathrm{d}(U) = m-1$) where $k \geq 1$ and $\forall 1 \leq i \leq k$, $M : \langle \Gamma, x^n : \vec{e}_{i(1:m-1)} V_i \vdash_2 \vec{e}_{i(1:m-1)} T_i\rangle$. By e, $\forall 1 \leq i \leq k$, $M^+ : \langle \Gamma, x^n + 1 : e\vec{e}_{i(1:m-1)} V_i \vdash_2 e\vec{e}_{i(1:m-1)} T_i\rangle$.

- Let $\dfrac{\lambda x^n.M : \langle \Gamma \vdash_2 U\rangle \quad \langle \Gamma \vdash_2 U\rangle \sqsubseteq \langle \Gamma' \vdash_2 U'\rangle}{\lambda x^n.M : \langle \Gamma' \vdash_2 U'\rangle}$. By lemma 21, $\Gamma' \sqsubseteq \Gamma$ and $U \sqsubseteq U'$. By lemma 23, U, U' are good and $\mathrm{d}(U) = \mathrm{d}(U') = m$. By IH, $U = \sqcap_{i=1}^k \vec{e}_{i(1:m)}(V_i \to T_i)$, where $k \geq 1$ and $M : \langle \Gamma, x^n : \vec{e}_{i(1:m)} V_i \vdash_2 \vec{e}_{i(1:m)} T_i\rangle$ $\forall\ 1 \leq i \leq k$. By lemma 21, $U' = \sqcap_{i=1}^p \vec{e'}_{i(1:m)}(V'_i \to T'_i)$, where $p \geq 1$, and $\forall 1 \leq i \leq p$, $\exists 1 \leq j_i \leq k$ such that $\vec{e}_{j_i(1:m)} = \vec{e'}_{i(1:m)}$, $V'_i \sqsubseteq V_{j_i}$ and $T_{j_i} \sqsubseteq T'_i$. Let $1 \leq i \leq p$. Since $\langle \Gamma, x^n : \vec{e}_{j_i(1:m)} V_{j_i} \vdash_2 \vec{e}_{j_i(1:m)} T_{j_i}\rangle \sqsubseteq \langle \Gamma', x^n : \vec{e'}_{i(1:m)} V'_i \vdash_2 \vec{e'}_{i(1:m)} T'_i\rangle$, by lemma 21, then $M : \langle \Gamma', x^n : \vec{e'}_{i(1:m)} V'_i \vdash_2 \vec{e'}_{i(1:m)} T'_i\rangle$.

□

The next lemma says that there are no blocked β-redexes in a typable term.

Lemma 27 (No β-redexes are blocked in typable terms) *Let $i \in \{1,2\}$ and $M : \langle \Gamma \vdash_i U\rangle$. If $(\lambda x^n.M_1)M_2$ is a subterm of M, then $d(M_2) = n$ and hence $(\lambda x^n.M_1)M_2 \rhd_\beta M_1[x^n := M_2]$.*

Proof

- Case $\vdash_1$. By induction on the derivation of $M : \langle \Gamma \vdash_1 U\rangle$. The only interesting case is $\to_e$ where $M = (\lambda x^n.M_1)M_2$ is the subterm in question. Here, $\dfrac{\lambda x^n.M_1 : \langle \Gamma_1 \vdash_1 T_1 \to T_2\rangle \quad M_2 : \langle \Gamma_2 \vdash T_1\rangle \quad \Gamma_1 \diamond \Gamma_2}{(\lambda x^n.M_1)M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_1 T_2\rangle}$.
  By Lemma 25.2, $M_1 : \langle \Gamma_1, x^n : T_1 \vdash_1 T_2\rangle$. By lemma 23, $n = \mathrm{d}(T_1)$ and $\mathrm{d}(M_2) = \mathrm{d}(T_1)$. Hence, $n = \mathrm{d}(M_2)$ and $(\lambda x^n.M_1)M_2 \rhd_\beta M_1[x^n := M_2]$.

- Case $\vdash_2$. By lemma 22.5, $(\lambda x^n.M_1)M_2$ is typable. By induction on the typing of $(\lambda x^n.M_1)M_2$. We consider only the rule $\to_e$:
  $\dfrac{\lambda x^n.M_1 : \langle \Gamma_1 \vdash' V \to T\rangle \quad M_2 : \langle \Gamma_2 \vdash' V\rangle \quad \Gamma_1 \diamond \Gamma_2}{(\lambda x^n.M_1)M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash' T\rangle}$.
  By lemma 12.2, $\mathrm{d}(V \to T) = 0$. By Lemma 26.2, $V \to T = \sqcap_{i=1}^k (V_i \to T_i)$ where $k \geq 1$ and $\forall 1 \leq i \leq k$, $M_1 : \langle \Gamma_1, x^n : V_i \vdash' T_i\rangle$. Hence $k = 1$, $V_i = V$, $T_i = T$ and $M_1 : \langle \Gamma_1, x^n : V \vdash' T\rangle$. By lemma 23, V is good, $\mathrm{d}(M_2) = \mathrm{d}(V)$ and $\mathrm{d}(V) = n$. So, $\mathrm{d}(M_2) = n$ and $(\lambda x^n.M_1)M_2 \rhd_\beta M_1[x^n := M_2]$.

□

4.2 Failure of subject reduction using $\vdash_1$

The next lemma shows that the substitution lemma for $\vdash_1$, and subject reduction for β using $\vdash_1$ fail. (See lemma 31 and corollary 33 for the statements of substitution and subject reduction.)

Lemma 28 (Subject β-reduction fails for $\vdash_1$) *Let a, b, c be different elements of $\mathcal{A}$. We have:*

1. $(\lambda x^0.x^0x^0)(y^0z^0) \rhd_\beta (y^0z^0)(y^0z^0)$

2. $(\lambda x^0.x^0x^0)(y^0z^0) : \langle y^0 : b \to ((a \to c) \sqcap a), z^0 : b \vdash_1 c\rangle$.

3. $x^0x^0 : \langle x^0 : (a \to c) \sqcap a \vdash_1 c\rangle$.

4. *It is not possible that*
$(y^0z^0)(y^0z^0) : \langle y^0 : b \to ((a \to c) \sqcap a), z^0 : b \vdash_1 c\rangle$.
Hence, the substitution and subject β-reduction lemmas fail for $\vdash_1$.

Proof 1..3 are easy. For 4, assume $(y^0z^0)(y^0z^0) : \langle y^0 : b \to ((a \to c) \sqcap a), z^0 : b \vdash_1 c\rangle$. By lemma 25.3 twice using lemma 22, lemmas 23 and 25.1:

- $y^0z^0 : \langle y^0 : b \to ((a \to c) \sqcap a), z^0 : b \vdash_1 \sqcap_{i=1}^n (T_i \to c)\rangle$.
- $y^0 : \langle y^0 : b \to ((a \to c) \sqcap a) \vdash_1 b \to (a \to c) \sqcap a\rangle$.
- $z^0 : \langle z^0 : b \vdash_1 b\rangle$.
- $\sqcap_{i=1}^n (T_i \to c) = (a \to c) \sqcap a$.

Hence $a = T_i \to c$ for some T_i. Absurd. □

4.3 Subject reduction and expansion using $\vdash_2$

In this section we show that the substitution lemma holds for $\vdash_2$ and we use this to show that subject reduction and subject expansion for β using $\vdash_2$ holds. The subject reduction and expansion for β will be used in the proof of completeness (more specifically in lemma 52 which is basic for the completeness theorem 53).

Just as we defined the degree decreasing of a term, we do the same for a type.

Definition 29 1. *If $d(U) > 0$, we inductively define the type U^- as follows:*
$(U_1 \sqcap U_2)^- = U_1^- \sqcap U_2^-$ $\qquad (eU)^- = U$

If $d(U) \geq n > 0$, we write U^{-n} for $(\dots(U\overbrace{^-)^- \dots {}^-)^-}^{n}$.

2. *If $\Gamma = (x_i^{n_i} : U_i)_k$ and $d(\Gamma) > 0$, then we let*
$\Gamma^- = (x_i^{n_i - 1} : U_i^-)_k$.

If $d(\Gamma) \geq n > 0$, we write Γ^{-n} for $(\dots(\Gamma\overbrace{^-)^- \dots {}^-)^-}^{n}$.

3. *If U is a type and Γ is a type environment such that $d(\Gamma) > 0$ and $d(U) > 0$, then we let $(\langle\Gamma \vdash_2 U\rangle)^- = (\langle\Gamma^- \vdash_2 U^-\rangle)$.*

Lemma 30 1. *If $d(U) > 0$, then $d(U^-) = d(U) - 1$.*

2. *If $d(U) > 0$ and U is a good type, then U^- is a good type.*

3. *If $d(U_1) > 0$ and $U_1 \sqsubseteq U_2$, then $U_1^- \sqsubseteq U_2^-$.*

4. *If $d(\Phi_1) > 0$ and $\Phi_1 \sqsubseteq \Phi_2$, then $\Phi_1^- \sqsubseteq \Phi_2^-$.*

5. *If $M : \Phi$ and $d(\Phi) > 0$, then $M^- : \Phi^-$.*

6. *If $M^+ : \langle\Gamma \vdash_2 eU\rangle$, then $M : \langle\Gamma^- \vdash_2 U\rangle$.*

Proof 1. By induction on U.
2. By induction on U using 1. and lemma 12.
3. By induction on the derivation of $U_1 \sqsubseteq U_2$ using 1, 2 and lemma 21.
4. Use 3 and lemma 21.
5. By induction on the derivation of $M : \Phi$. By lemma 12.2, we have three cases.

- Let $\dfrac{M : \langle \Gamma \vdash_2 W_1 \rangle \quad M : \langle \Gamma \vdash_2 W_2 \rangle}{M : \langle \Gamma \vdash_2 W_1 \sqcap W_2 \rangle}$. By lemma 23, $d(W_1) = d(W_2) = d(W_1 \sqcap W_2) > 0$. By IH, $M^- : \langle \Gamma^- \vdash_2 W_1^- \rangle$ and $M^- : \langle \Gamma^- \vdash_2 W_2^- \rangle$. Hence, by $\sqcap_i$, $M^- : \langle \Gamma^- \vdash_2 W_1^- \sqcap W_2^- \rangle$.

- Let $\dfrac{M : \langle (x_i^{n_i} : U_i)_n \vdash_2 U \rangle}{M^+ : \langle (x_i^{n_i+1} : eU_i)_n \vdash_2 eU \rangle}$. By lemma 8.1a, $(M^+)^- = M$.

- Let $\dfrac{M : \Phi_1 \quad \Phi_1 \sqsubseteq \Phi_2}{M : \Phi_2}$. By lemma 21, $d(\Phi_1) > 0$. Hence, by IH, $M^- : \Phi_1^-$. By 4, $\Phi_1^- \sqsubseteq \Phi_1^-$. Hence, by $\sqsubseteq$, $M^- : \Phi_2^-$.

6. Note that $d(eU) > 0$. Hence, by lemma 5, $(M^+)^- : \langle \Gamma^- \vdash_2 (eU)^- \rangle$. Hence, by lemma 8.1a, $M : \langle \Gamma^- \vdash_2 U \rangle$. □

The next lemma which fails for $\vdash_1$, is needed in the proof of subject reduction for β using $\vdash_2$.

Lemma 31 (Substitution for $\vdash_2$) *If $M : \langle \Gamma, x^n : U \vdash_2 V \rangle$, $N : \langle \Delta \vdash_2 U \rangle$ and $\Gamma \diamond \Delta$, then $M[x^n := N] : \langle \Gamma \sqcap \Delta \vdash_2 V \rangle$.*

Proof By induction on the derivation of $M : \langle \Gamma, x^n : U \vdash_2 V \rangle$.

- If $\dfrac{T \textit{ good}}{x^0 : \langle (x^0 : T) \vdash_2 T \rangle}$ and $N : \langle \Delta \vdash_2 T \rangle$, then $N = x^0[x^0 := N] : \langle \Delta \vdash_2 T \rangle$.

- Let $\dfrac{M : \langle \Gamma, x^n : U, y^m : U' \vdash_2 T \rangle}{\lambda y^m.M : \langle \Gamma, x^n : U \vdash_2 U' \to T \rangle}$. Since $\Gamma \diamond \Delta$, by BC, $\Gamma, y^m : U' \diamond \Delta$ and $y^m \notin dom(\Delta)$. By IH, $M[x^n := N] : \langle \Gamma \sqcap \Delta, y^m : U' \vdash_2 T \rangle$. By $\to_i$, $(\lambda y^m.M)[x^n := N] = \lambda y^m.M[x^n := N] : \langle \Gamma \sqcap \Delta \vdash_2 U' \to T \rangle$.

- Let $\dfrac{M_1 : \langle \Gamma_1, x^n : U_1 \vdash_2 V \to T \rangle \quad M_2 : \langle \Gamma_2, x^n : U_2 \vdash_2 V \rangle \quad \Gamma_1 \diamond \Gamma_2}{M_1 M_2 : \langle \Gamma_1 \sqcap \Gamma_2, x^n : U_1 \sqcap U_2 \vdash_2 T \rangle}$ where $x^n \in FV(M_1) \cap FV(M_2)$, $N : \langle \Delta \vdash_2 U_1 \sqcap U_2 \rangle$ and $(\Gamma_1 \sqcap \Gamma_2) \diamond \Delta$. By $\sqcap_e$ and $\sqsubseteq$, $N : \langle \Delta \vdash_2 U_1 \rangle$ and $N : \langle \Delta \vdash_2 U_2 \rangle$. Now use IH and $\to_e$.
  The cases $x^n \in FV(M_1) \setminus FV(M_2)$ or $x^n \in FV(M_2) \setminus FV(M_1)$ are easy.

- If $\dfrac{M : \langle \Gamma, x^n : U \vdash_2 U_1 \rangle \quad M : \langle \Gamma, x^n : U \vdash_2 U_2 \rangle}{M : \langle \Gamma, x^n : U \vdash_2 U_1 \sqcap U_2 \rangle}$ use IH and $\sqcap_i$.

- Let $\dfrac{M : \langle \Gamma, x^n : U \vdash_2 V \rangle}{M^+ : \langle e\Gamma, x^{n+1} : eU \vdash_2 eV \rangle}$ where $N : \langle \Delta \vdash_2 eU \rangle$ and $e\Gamma \diamond \Delta$. By lemma 23, $d(N) = d(eU) = d(U) + 1 > 0$. Hence, by lemmas 8.3 and 30.6, $N = P+$ and $P : \langle \Delta^- \vdash_2 U \rangle$. As $e\Gamma \diamond \Delta$, then $\Gamma \diamond \Delta^-$. By IH, $M[x^n := P] : \langle \Gamma \sqcap \Delta^- \vdash_2 V \rangle$. By e and lemma 8.2, $M^+[x^{n+1} := N] : \langle e\Gamma \sqcap \Delta \vdash_2 eV \rangle$.

- Let $\dfrac{M : \langle \Gamma', x^n : U' \vdash_2 V' \rangle \quad \langle \Gamma', x^n : U' \vdash_2 V' \rangle \sqsubseteq \langle \Gamma, x^n : U \vdash_2 V \rangle}{M : \langle \Gamma, x^n : U \vdash_2 V \rangle}$ (note the use of lemma 21). By lemma 21, $dom(\Gamma) = dom(\Gamma')$, $\Gamma \subseteq \Gamma'$, $U \subseteq U'$ and $V' \subseteq V$. Hence $\Gamma' \diamond \Delta$, $N : \langle \Delta \vdash_2 U' \rangle$ and, by IH, $M[x^n := N] : \langle \Gamma' \sqcap \Delta \vdash_2 V' \rangle$. It is easy to show that $\Gamma \sqcap \Delta \subseteq \Gamma' \sqcap \Delta$. Hence, $\langle \Gamma' \sqcap \Delta \vdash_2 V' \rangle \subseteq \langle \Gamma \sqcap \Delta \vdash_2 V \rangle$ and $M[x^n := N] : \langle \Gamma \sqcap \Delta \vdash_2 V \rangle$.

□

Now, we give the basic block in the proof of subject reduction for β.

Theorem 32 *If $M : \langle \Gamma \vdash_2 U\rangle$ and $M \rhd_\beta N$, then $N : \langle \Gamma \vdash_2 U\rangle$.*

Proof By induction on the derivation of $M : \langle \Gamma \vdash_2 U\rangle$. $\to_i$, $\sqcap_i$ and $\sqsubseteq$ are by IH. We give the remaining two cases.

- Let $\dfrac{M_1 : \langle \Gamma_1 \vdash_2 U \to T\rangle \quad M_2 : \langle \Gamma_2 \vdash_2 U\rangle \quad \Gamma_1 \diamond \Gamma_2}{M_1M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_2 T\rangle}$. For the cases $N = M_1N_2$ where $M_2 \rhd_\beta N_2$ or $N = N_1M_2$ where $M_1 \rhd_\beta N_1$ use IH. Assume $M_1 = \lambda x^n.P$ and $M_1M_2 = (\lambda x^n.P)M_2 \rhd_\beta P[x^n := M_2] = N$ where $\mathrm{d}(M_2) = n$. Since $\lambda x^n.P : \langle \Gamma_1 \vdash_2 U \to T\rangle$ and, by lemma 12.2.2a $\mathrm{d}(U \to T) = 0$, then, by lemma 26.2, $P : \langle \Gamma_1, x^n : U \vdash_2 T\rangle$. By lemma 31, $P[x^n := M_2] : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_2 T\rangle$.

- Let $\dfrac{M : \langle \Gamma \vdash_2 U\rangle}{M^+ : \langle e\Gamma \vdash_2 eU\rangle}$. If $M^+ \rhd_\beta N$, then by lemma 6.1, $\mathrm{d}(M^+) = \mathrm{d}(N)$. By lemmas 8.1a and 8.3, $\mathrm{d}(N) > 0$, $N = P^+$ and $M \rhd_\beta P$. By IH, $P : \langle \Gamma \vdash_2 U\rangle$ and, by *exp*, $N : \langle e\Gamma \vdash_2 eU\rangle$.

$\square$

Corollary 33 (Subject reduction for β)
If $M : \langle \Gamma \vdash_2 U\rangle$ and $M \rhd_\beta^* N$, then $N : \langle \Gamma \vdash_2 U\rangle$.

Proof By induction on the length of the derivation of $M \rhd_\beta^* N$ using theorem 32. $\square$

The next lemma will be used in the proof of subject expansion for β.

Lemma 34 *Let $(\lambda x^n.M_1)M_2 : \langle \Gamma \vdash_2 U\rangle$ then $\Gamma = \Gamma_1 \sqcap \Gamma_2$ and $\exists V \in \mathbb{U}$ such that $M_1 : \langle \Gamma_1, (x^n : V) \vdash_2 U\rangle$ and $M_2 : \langle \Gamma_2 \vdash_2 V\rangle$.*

Proof By induction on the derivation of $(\lambda x^n.M_1)M_2 : \langle \Gamma \vdash_2 U\rangle$.

- Let $\dfrac{\lambda x^n.M_1 : \langle \Gamma_1 \vdash_2 V \to T\rangle \quad M_2 : \langle \Gamma_2 \vdash_2 V\rangle \quad \Gamma_1 \diamond \Gamma_2}{(\lambda x^n.M_1)M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_2 T\rangle}$.
Since $\mathrm{d}(V \to T) = 0$, by lemma 26.2 $M_1 : \langle \Gamma_1, (x^n : V) \vdash_2 T\rangle$.

- Let $\dfrac{(\lambda x^n.M_1)M_2 : \langle \Gamma \vdash_2 U_1\rangle \quad (\lambda x^n.M_1)M_2 : \langle \Gamma \vdash_2 U_2\rangle}{(\lambda x^n.M_1)M_2 : \langle \Gamma \vdash_2 U_1 \sqcap U_2\rangle}$. By IH, $\Gamma = \Gamma_1 \sqcap \Gamma_2 = \Gamma'_1 \sqcap \Gamma'_2$, $\exists V, V' \in \mathbb{U}$, such that $M_1 : \langle \Gamma_1, (x^n : V) \vdash_2 U_1\rangle$, $M_2 : \langle \Gamma_2 \vdash_2 V\rangle$, $M_1 : \langle \Gamma'_1, (x^n : V') \vdash_2 U_2\rangle$ and $M_2 : \langle \Gamma'_2 \vdash_2 V'\rangle$. By lemma 23.1, $\Gamma_1, \Gamma_2, \Gamma'_1$, Γ'_2, V and V' are all good. By lemma 22.1, $dom(\Gamma_1, (x^n : V)) = FV(M_1) = dom(\Gamma'_1, (x^n : V'))$ so $dom(\Gamma_1) = dom(\Gamma'_1)$ and $dom(\Gamma_2) = FV(M_2) = dom(\Gamma'_2)$. Hence,by $\sqcap_e$, and lemma 21, $\Gamma_1 \sqcap \Gamma'_1, (x^n : V \sqcap V') \sqsubseteq \Gamma_1, (x^n : V)$, $\Gamma_1 \sqcap \Gamma'_1, (x^n : V \sqcap V') \sqsubseteq \Gamma'_1, (x^n : V')$, $\Gamma_2 \sqcap \Gamma'_2 \sqsubseteq \Gamma_2$ and $\Gamma_2 \sqcap \Gamma'_2 \sqsubseteq \Gamma'_2$. By lemma 21.3 and $\sqsubseteq$, $M_1 : \langle \Gamma_1 \sqcap \Gamma'_1, (x^n : V \sqcap V') \vdash_2 U_1\rangle$, $M_1 : \langle \Gamma_1 \sqcap \Gamma'_1, (x^n : V \sqcap V') \vdash_2 U_2\rangle$, $M_2 : \langle \Gamma_2 \sqcap \Gamma'_2 \vdash_2 V\rangle$ and $M_2 : \langle \Gamma_2 \sqcap \Gamma'_2 \vdash_2 V'\rangle$. So by $\sqcap_i$, $M_1 : \langle \Gamma_1 \sqcap \Gamma'_1, (x^n : V \sqcap V') \vdash_2 U_1 \sqcap U_2\rangle$ and $M_2 : \langle \Gamma_2 \sqcap \Gamma'_2 \vdash_2 V \sqcap V'\rangle$.

- Let $\dfrac{(\lambda x^n.M_1)M_2 : \langle \Gamma \vdash_2 U\rangle}{(\lambda x^{n+1}.M_1^+)M_2^+ : \langle e\Gamma \vdash_2 eU\rangle}$. By IH, $\Gamma = \Gamma_1 \sqcap \Gamma_2$ and $\exists V \in \mathbb{U}$, such that $M_1 : \langle \Gamma_1, (x^n : V) \vdash_2 U\rangle$ and $M_2 : \langle \Gamma_2 \vdash_2 V\rangle$. So by *exp*, $M_1^+ : \langle e\Gamma_1, (x^{n+1} : eV) \vdash_2 eU\rangle$ and $M_2^+ : \langle e\Gamma_2 \vdash_2 eV\rangle$.

- Let $\dfrac{(\lambda x^n.M_1)M_2 : \langle \Gamma' \vdash_2 U'\rangle \quad \langle \Gamma' \vdash_2 U'\rangle \sqsubseteq \langle \Gamma \vdash_2 U\rangle}{(\lambda x^n.M_1)M_2 : \langle \Gamma \vdash_2 U\rangle}$. By lemma 21.3, $\Gamma \sqsubseteq \Gamma'$ and $U' \sqsubseteq U$. By IH, $\Gamma' = \Gamma'_1 \sqcap \Gamma'_2$ and $\exists V \in \mathbb{U}$, such that $M_1 : \langle \Gamma'_1, (x^n : V) \vdash_2 U'\rangle$ and $M_2 : \langle \Gamma'_2 \vdash_2 V\rangle$. By lemma 21.11, $\Gamma = \Gamma_1 \sqcap \Gamma_2$ such that $\Gamma_1 \sqsubseteq \Gamma'_1$ and $\Gamma_2 \sqsubseteq \Gamma'_2$. So by $\sqsubseteq$, $M_1 : \langle \Gamma_1, (x^n : V) \vdash_2 U\rangle$ and $M_2 : \langle \Gamma_2 \vdash_2 V\rangle$.

$\square$

Now, we give the basic block in the proof of subject expansion for β.

Lemma 35 *If $N : \langle\Gamma \vdash_2 U\rangle$ and $M \rhd_\beta N$ then $M : \langle\Gamma \vdash_2 U\rangle$*

Proof By induction on the derivation of $N : \langle\Gamma \vdash_2 U\rangle$.

- Let $\dfrac{T\ good}{x^0 : \langle(x^0 : T) \vdash_2 T\rangle}$ where $M \rhd_\beta x^0$. By cases on M, we can show that $M = (\lambda y^0.y^0)x^0$. Since T is good, by ax, $y^0 : \langle(y^0 : T) \vdash_2 T\rangle$, then by $\to_i$, $\lambda y^0.y^0 : \langle() \vdash_2 T \to T\rangle$, and so by $\to_e$, $(\lambda y^0.y^0)x^0 : \langle(x^0 : T) \vdash_2 T\rangle$.

- Let $\dfrac{N : \langle\Gamma, (x^n : U) \vdash_2 T\rangle}{\lambda x^n.N : \langle\Gamma \vdash_2 U \to T\rangle}$ where $M \rhd_\beta \lambda x^n.N$. By cases on M.

  - If M is a variable this is not possible.
  - If $M = \lambda x^n.M'$ such that $M' \rhd_\beta N$ and $x^n \in FV(M') \cap FV(N)$ then by IH, $M : \langle\Gamma, (x^n : U) \vdash_2 T\rangle$ and by $\to_i$, $M : \langle\Gamma \vdash_2 U \to T\rangle$.
  - If M is an application term then the reduction must be at the root. Hence, $M = (\lambda y^m.M_1)M_2 \rhd_\beta M_1[y^m := M_2] = \lambda x^n.N$ where $y^m \in FV(M_1)$. There are two cases (M_1 cannot be an application term):
    - If $M_1 = y^m$ then $M_2 = \lambda x^n.N$ and $\mathrm{d}(N) = m$. By lemma 23.2, $m = \mathrm{d}(N) = \mathrm{d}(T) = 0$. So $M = (\lambda y^0.y^0)(\lambda x^n.N)$. Since by lemma 23.2, $U \to T$ is good , by ax, $y^0 : \langle(y^0 : U \to T) \vdash_2 U \to T\rangle$, then by $\to_i$, $\lambda y^0.y^0 : \langle() \vdash_2 (U \to T) \to (U \to T)\rangle$, and so by $\to_e$, $(\lambda y^0.y^0)(\lambda x^n.N) : \langle\Gamma \vdash_2 U \to T\rangle$.
    - If $M_1 = \lambda x^n.M_1'$ then $M_1[y^m := M_2] = \lambda x^n.M_1'[y^m := M_2] = \lambda x^n.N$ and $\mathrm{d}(M_2) = m$. Since $(\lambda y^m.M_1')M_2 \rhd_\beta M_1'[y^m := M_2] = N$, by IH, $(\lambda y^m.M_1')M_2 : \langle\Gamma, (x^n : U) \vdash_2 T\rangle$. By lemma 34, $\Gamma, (x^n : U) = \Gamma_1 \sqcap \Gamma_2$ and $\exists V \in \mathbb{U}$ such that $M_1' : \langle\Gamma_1, (y^m : V) \vdash_2 T\rangle$ and $M_2 : \langle\Gamma_2 \vdash_2 V\rangle$. Since $M \in \mathcal{M}$, $y^m \in FV(M_1')$ and so (since $x^n \notin FV(M_2)$), by lemma 22 $\Gamma = \Gamma_1' \sqcap \Gamma_2$ and $\Gamma_1 = \Gamma_1', (x^n : U)$. Hence by $\to_i$, $\lambda x^n.M_1' : \langle\Gamma_1', (y^m : V) \vdash_2 U \to T\rangle$, again by $\to_i$, $\lambda y^m.\lambda x^n.M_1' : \langle\Gamma_1' \vdash_2 V \to U \to T\rangle$, and since by lemma 22.6, $\Gamma_1' \diamond \Gamma_2$, by $\to_e$, $M = (\lambda y^m.\lambda x^n.M_1')M_2 : \langle\Gamma \vdash_2 U \to T\rangle$.

- Let $\dfrac{N_1 : \langle\Gamma_1 \vdash_2 U \to T\rangle \quad N_2 : \langle\Gamma_2 \vdash_2 U\rangle \qquad \Gamma_1 \diamond \Gamma_2}{N_1N_2 : \langle\Gamma_1 \sqcap \Gamma_2 \vdash_2 T\rangle}$ and $M \rhd_\beta N_1N_2$.

  - If $M = M_1N_2 \rhd_\beta N_1N_2$ where $M_1 \diamond N_2$, $N_1 \diamond N_2$ and $M_1 \rhd_\beta N_1$ then by IH, $M_1 : \langle\Gamma_1 \vdash_2 U \to T\rangle$, and by $\to_e$, $M : \langle\Gamma_1 \sqcap \Gamma_2 \vdash_2 T\rangle$.
  - If $M = N_1M_2 \rhd_\beta N_1N_2$ where $N_1 \diamond M_2$, $N_1 \diamond N_2$ and $M_2 \rhd_\beta N_2$ then by IH, $M_2 : \langle\Gamma_2 \vdash_2 U\rangle$, and by $\to_e$, $M : \langle\Gamma_1 \sqcap \Gamma_2 \vdash_2 T\rangle$.
  - If $M = (\lambda x^n.M_1)M_2 \rhd_\beta M_1[x^n := M_2] = N_1N_2$ where $\mathrm{d}(M_2) = n$ and $x^n \in FV(M_1)$. By cases on M_1 (M_1 cannot be an abstraction):
    - If $M_1 = x^n$ then $M_2 = N_1N_2$, $\mathrm{d}(N_1N_2) = n$ and $M = (\lambda x^0.x^0)(N_1N_2)$. By lemma 23, $n = 0$ and T is good. By ax, $x^0 : \langle(x^0 : T) \vdash_2 T\rangle$, hence by $\to_i$, $\lambda x^0.x^0 : \langle() \vdash_2 T \to T\rangle$, and by $\to_e$, $(\lambda x^0.x^0)(N_1N_2) : \langle\Gamma_1 \sqcap \Gamma_2 \vdash_2 T\rangle$.
    - If $M_1 = M_1'M_1''$ then $M_1[x^n := M_2] = M_1'[x^n := M_2]M_1''[x^n := M_2] = N_1N_2$. So, $M_1'[x^n := M_2] = N_1$ and $M_1''[x^n := M_2] = N_2$.
      - If $x^n \in FV(M_1')$ and $x^n \in FV(M_1'')$ then $(\lambda x^n.M_1')M_2 \rhd_\beta N_1$ and $(\lambda x^n.M_1'')M_2 \rhd_\beta N_2$. By IH, $(\lambda x^n.M_1')M_2 : \langle\Gamma_1 \vdash_2 U \to T\rangle$ and $(\lambda x^n.M_1'')M_2 : \langle\Gamma_2 \vdash_2 U\rangle$. By lemma 34 twice, $\Gamma_1 = \Gamma_1' \sqcap \Gamma_1''$, $\Gamma_2 = \Gamma_2' \sqcap \Gamma_2''$, and $\exists V, V' \in \mathbb{U}$ such that $M_1' : \langle\Gamma_1', (x^n : V) \vdash_2$

$U \to T\rangle$, $M_2 : \langle\Gamma_1'' \vdash_2 V\rangle$, $M_1'' : \langle\Gamma_2', (x^n : V') \vdash_2 U\rangle$ and $M_2 : \langle\Gamma_2'' \vdash_2 V'\rangle$. By lemma 23.1, $\Gamma_1', \Gamma_2', \Gamma_1''$, Γ_2'', V and V' are all good. By lemma 22.1, $dom(\Gamma_1'') = FV(M_2) = dom(\Gamma_2'')$. Hence, by $\sqcap_e$, lemma 21, $\sqsubseteq$ and $\sqcap_i$, $M_2 : \langle\Gamma_1'' \sqcap \Gamma_2'' \vdash_2 V \sqcap V'\rangle$. Since by lemma 22.6, $\Gamma_1' \diamond \Gamma_2'$, by $\to_e$, $M_1'M_1'' : \langle\Gamma_1' \sqcap \Gamma_2', (x^n : V \sqcap V') \vdash_2 T\rangle$. So by $\to_i$, $\lambda x^n.M_1'M_1'' : \langle\Gamma_1' \sqcap \Gamma_2' \vdash_2 (V \sqcap V') \to T\rangle$. Finally, by $\to_e$ and since by lemma 22.6, $\Gamma_1' \sqcap \Gamma_2' \diamond \Gamma_1'' \sqcap \Gamma_2''$ and $\Gamma_1 \sqcap \Gamma_2 = \Gamma_1' \sqcap \Gamma_2' \sqcap \Gamma_1'' \sqcap \Gamma_2''$, $(\lambda x^n.M_1'M_1'')M_2 : \langle\Gamma_1 \sqcap \Gamma_2 \vdash_2 T\rangle$.

· If $x^n \in FV(M_1')$ and $x^n \notin FV(M_1'')$ then $M_1'[x^n := M_2] = N_1$ and $M_1'' = N_2$. We have $(\lambda x^n.M_1')M_2 \rhd_\beta N_1$, so by IH, $(\lambda x^n.M_1')M_2 : \langle\Gamma_1 \vdash_2 U \to T\rangle$. By lemma 34, $\Gamma_1 = \Gamma_1' \sqcap \Gamma_1''$ and $\exists V \in \mathbb{U}$ such that $M_1' : \langle\Gamma_1', (x^n : V) \vdash_2 U \to T\rangle$ and $M_2 : \langle\Gamma_1'' \vdash_2 V\rangle$. Since by lemma 22.6, $\Gamma_1' \diamond \Gamma_2$, by $\to_e$, $M_1'M_1'' : \langle\Gamma_1' \sqcap \Gamma_2, (x^n : V) \vdash_2 T\rangle$, and by $\to_i$, $\lambda x^n.M_1'M_1'' : \langle\Gamma_1' \sqcap \Gamma_2 \vdash_2 V \to T\rangle$. Finally, by $\to_e$ and since by lemma 22.6, $\Gamma_1' \sqcap \Gamma_2 \diamond \Gamma_1''$, $(\lambda x^n.M_1'M_1'')M_2 : \langle\Gamma_1 \sqcap \Gamma_2 \vdash_2 T\rangle$.

· If $x^n \notin FV(M_1')$ and $x^n \in FV(M_1'')$ then the proof is similar to the previous case.

- Let $\dfrac{N : \langle\Gamma \vdash_2 U_1\rangle \quad N : \langle\Gamma \vdash_2 U_2\rangle}{N : \langle\Gamma \vdash_2 U_1 \sqcap U_2\rangle}$ and $M \rhd_\beta N$. By IH, $M : \langle\Gamma \vdash_2 U_1\rangle$ and $M : \langle\Gamma \vdash_2 U_2\rangle$, hence by $\sqcap_i$, $M : \langle\Gamma \vdash_2 U_1 \sqcap U_2\rangle$.

- Let $\dfrac{N : \langle\Gamma \vdash_2 U\rangle}{N^+ : \langle e\Gamma \vdash_2 eU\rangle}$ and $M \rhd_\beta N^+$. By lemma 8.6, $M^- \rhd_\beta N$, and by IH, $M^- : \langle\Gamma \vdash_2 U\rangle$. By lemma 8.1b, $(M^-)^+ = M$ and by *exp*, $M : \langle e\Gamma \vdash_2 eU\rangle$.

- Let $\dfrac{N : \langle\Gamma \vdash_2 U\rangle \quad \langle\Gamma \vdash_2 U\rangle \sqsubseteq \langle\Gamma' \vdash_2 U'\rangle}{N : \langle\Gamma' \vdash_2 U'\rangle}$ and $M \rhd_\beta N$. By IH, $M : \langle\Gamma \vdash_2 U\rangle$ and by $\sqsubseteq$ $M : \langle\Gamma' \vdash_2 U'\rangle$.

□

Corollary 36 (Subject expansion for β)
If $N : \langle\Gamma \vdash_2 U\rangle$ and $M \rhd_\beta^* N$ then $M : \langle\Gamma \vdash_2 U\rangle$

Proof By induction on the length of the derivation of $M \rhd_\beta^* N$ using lemma 35. □

5 Soundness of the realisability semantics for $\vdash_1/\vdash_2$ and examples

Lemma 37 *If $\mathcal{I}$ be an interpretation and $U \sqsubseteq V$, then $\mathcal{I}(U) \subseteq \mathcal{I}(V)$.*

Proof By induction of the derivation of $U \sqsubseteq V$. □

We already gave the realisability semantics for the types in $\mathcal{T}$ and $\mathbb{U}$ in section 3.2. The next lemma shows that this semantics is sound with respect to $\vdash_1$ and $\vdash_2$.

Lemma 38 (Soundness of $\vdash_1/\vdash_2$) *Let $i \in \{1, 2\}$ and $M : \langle(x_i^{n_i} : U_i)_n \vdash_i U\rangle$, $\mathcal{I}$ be an interpretation and $\forall 1 \le i \le n$, $N_i \in \mathcal{I}(U_i)$. If $M[(x_i^{n_i} := N_i)_n] \in \mathcal{M}$, then $M[(x_i^{n_i} := N_i)_n] \in \mathcal{I}(U)$.*

Proof By induction on the derivation of $M : \langle(x_i^{n_i} : U_i)_n \vdash_i U\rangle$. First note, by lemma 23 and lemma 17, $\forall 1 \le i \le n$, U_i is good and $N_i \in \mathcal{M}$.

- If $\dfrac{T \ good \quad \mathrm{d}(T) = n}{x^n : \langle(x^n : T) \vdash_i T\rangle}$ and $N \in \mathcal{I}(T)$, then $x^n[x^n := N] = N \in \mathcal{I}(T)$.

- Let $\dfrac{M : \langle (x_i^{n_i} : U_i)_n, (x^m : U) \vdash_i T\rangle}{\lambda x^m.M : \langle (x_i^{n_i} : U_i)_n \vdash_i U \to T\rangle}$ and $\forall 1 \leq i \leq n$, $N_i \in \mathcal{I}(U_i)$ where $(\lambda x^m.M)[(x_i^{n_i} := N_i)_n] \in \mathcal{M}$. Let $N \in \mathcal{I}(U)$ where $(\lambda x^m.M)[(x_i^{n_i} := N_i)_n] \diamond N$. Since $(\lambda x^m.M[(x_i^{n_i} := N_i)_n]) \diamond N$, by lemma 3, $M[(x_i^{n_i} := N_i)_n] \diamond N$ and $M[(x_i^{n_i} := N_i)_n][x^m := N] = M[(x_i^{n_i} := N_i)_n, x^m := N] \in \mathcal{M}$. Hence, by IH, $M[(x_i^{n_i} := N_i)_n, x^m := N] \in \mathcal{I}(T)$. By lemma 23, U, T are good and $\mathrm{d}(U) = m$. By lemma 17, $\mathrm{d}(N) = m$ and $(\lambda x^m.M[(x_1^{n_1} := N_1)_n])N \rhd_\beta M[(x_i^{n_i} := N_i)_n, x^m := N] \in \mathcal{I}(T)$. Since, by lemma 17 $\mathcal{I}(T)$ is saturated, then $(\lambda x^m.M[(x_1^{n_1} := N_1)_n])N \in \mathcal{I}(T)$ and hence $\lambda x^m.M[(x_i^{n_i} := N_i)_n] \in \mathcal{I}(U) \rightsquigarrow \mathcal{I}(T) = \mathcal{I}(U \to T)$.

- Let $\dfrac{M_1 : \langle \Gamma_1 \vdash_i U \to T\rangle \quad M_2 : \langle \Gamma_2 \vdash_i U\rangle \quad \Gamma_1 \diamond \Gamma_2}{M_1\ M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_i T\rangle}$ where
  $\Gamma_1 = (x_i^{n_i} : U_i)_n, (y_j^{m_j} : V_j)_m$, $\Gamma_2 = (x_i^{n_i} : U'_i)_n, (z_k^{r_k} : W_k)_r$ and
  $\Gamma_1 \sqcap \Gamma_2 = (x_i^{n_i} : U_i \sqcap U'_i)_n, (y_j^{m_j} : V_j)_m, (z_k^{r_k} : W_k)_r$.

  Let $\forall\ 1 \leq i \leq n, P_i \in \mathcal{I}(U_i \sqcap U'_i)$, $\forall\ 1 \leq j \leq m, Q_j \in \mathcal{I}(V_j)$ and $\forall\ 1 \leq k \leq r, R_k \in \mathcal{I}(W_k)$ where
  $(M_1 M_2)[(x_i^{n_i} := P_i)_n, (y_j^{m_j} := Q_j)_m, (z_k^{r_k} := R_k)_r] \in \mathcal{M}$.
  Let $A = M_1[(x_i^{n_i} := P_i)_n, (y_j^{m_j} := Q_j)_m]$ and
  $B = M_2[(x_i^{n_i} := P_i)_n, (z_k^{r_k} := R_k)_r]$.

  By lemma 22, $FV(M_1) = dom(\Gamma_1)$ and $FV(M_2) = dom(\Gamma_2)$. Hence,
  $(M_1 M_2)[(x_i^{n_i} := P_i)_n, (y_j^{m_j} := Q_j)_m, (z_k^{r_k} := R_k)_r] = AB$.
  By lemma 3, $A \in \mathcal{M}$, $B \in \mathcal{M}$, and $A \diamond B$.
  By IH, $A \in \mathcal{I}(U) \rightsquigarrow \mathcal{I}(T)$ and $B \in \mathcal{I}(U)$.
  Hence, $AB = (M_1 M_2)[(x_i^{n_i} := P_i)_n, (y_j^{m_j} := Q_j)_m, (z_k^{r_k} := R_k)_r] \in \mathcal{I}(T)$.

- Let $\dfrac{M : \langle (x_i^{n_i} : U_i)_n \vdash_i U\rangle \quad M : \langle (x_i^{n_i} : V_i)_n \vdash_i V\rangle}{M : \langle (x_i^{n_i} : U_i \sqcap V_i)_n \vdash_i U \sqcap V\rangle}$ (note lemma 22.1) and $\forall\, 1 \leq i \leq n$, $N_i \in \mathcal{I}(U_i \sqcap V_i) = \mathcal{I}(U_i) \cap \mathcal{I}(V_i)$ where $M[(x_i^{n_i} := N_i)_n] \in \mathcal{M}$. By IH, $M[(x_i^{n_i} := N_i)_n] \in \mathcal{I}(U)$ and $M[(x_i^{n_i} := N_i)_n] \in \mathcal{I}(V)$. Hence, $M[(x_i^{n_i} := N_i)_n] \in \mathcal{I}(U \sqcap V)$.

- Let $\dfrac{M : \langle (x_i^{n_i} : T_i)_n \vdash_i U\rangle}{M^+ : \langle (x_i^{n_i+1} : eT_i)_n \vdash_i eU\rangle}$ and $\forall\ 1 \leq i \leq n$, $N_i \in \mathcal{I}(eT_i) = \mathcal{I}(T_i)^+$ where $M^+[(x_i^{n_i+1} := N_i)_n] \in \mathcal{M}$. Then $\forall\ 1 \leq i \leq n$, $N_i = P_i^+$ where $P_i \in \mathcal{I}(T_i)$. By lemmas 3 and 8.1(c)i, $\diamond\{M^+, N_1, \dots, N_n\}$ and $\diamond\{M, P_1, \dots, P_n\}$. Then, by lemma 3, $M[(x_i^{n_i} := P_i)_n] \in \mathcal{M}$ and, by IH, $M[(x_i^{n_i} := P_i)_n] \in \mathcal{I}(U)$. Hence, by lemma 8.2, $M^+[(x_i^{n_i+1} := P_i^+)_n] = (M[(x_i^{n_i} := P_i)_n])^+ \in \mathcal{I}(U)^+ = \mathcal{I}(eU)$.

- Let $\dfrac{M : \Phi \quad \Phi \sqsubseteq \Phi'}{M : \Phi'}$ where $\phi' = \langle (x_i^{n_i} : U_i)_n \vdash_2 U\rangle$. By lemma 21, we have $\Phi = \langle (x_i^{n_i} : U'_i)_n \vdash_2 U'\rangle$, where for every $1 \leq i \leq m$, $U_i \sqsubseteq U'_i$ and $U' \sqsubseteq U$. By lemma 37, $N_i \in \mathcal{I}(U'_i)$, then, by IH, $M[(x_i^{n_i} := N_i)_n] \in \mathcal{I}(U')$ and, by lemma 37, $M[(x_i^{n_i} := N_i)_n] \in \mathcal{I}(U)$.

$\square$

Theorem 39 (Soundness of $\vdash_1/\vdash_2$ for closed terms) *If $M : \langle () \vdash_i U\rangle$, then $M \in [U]$.*

Proof By lemma 38, $M \in \mathcal{I}(U)$ for any interpretation $\mathcal{I}$. By lemma 22, $FV(M) = dom(()) = \emptyset$ and hence M is closed. Therefore, $M \in [U]$. $\square$

The next definition and lemma put the realisability semantics in use.

Definition 40 (Examples) *Let $a, b \in \mathcal{A}$ where $a \neq b$. We define:*

- $Id_0 = a \to a$, $Id_1 = e(a \to a)$ *and* $Id_1' = ea \to ea$.
- $D = (a \sqcap (a \to b)) \to b$.
- $Nat_0 = (a \to a) \to (a \to a)$, $Nat_1 = e((a \to a) \to (a \to a))$, $Nat_1' = e(a \to a) \to (ea \to ea)$ *and* $Nat_0' = (ea \to a) \to (ea \to a)$.

Moreover, if M, N are terms and $n \in \mathbb{N}$, we define $(M)^n\ N$ by induction on n: $(M)^0\ N = N$ and $(M)^{m+1}\ N = M\ ((M)^m\ N)$.

Lemma 41

1. $[Id_0] = \{M \in \mathbb{M}^0\ /\ M \rhd_\beta^* \lambda y^0.y^0\}$.
2. $[Id_1] = [Id_1'] = \{M \in \mathbb{M}^1\ /\ M \rhd_\beta^* \lambda y^1.y^1\}$. *(Note that $Id_1' \notin \mathbb{U}$.)*
3. $[D] = \{M \in \mathbb{M}^0\ /\ M \rhd_\beta^* \lambda y^0.y^0 y^0\}$.
4. $[Nat_0] = \{M \in \mathbb{M}^0\ /\ M \rhd_\beta^* \lambda f^0.f^0$ *or* $M \rhd_\beta^* \lambda f^0.\lambda y^0.(f^0)^n y^0$ *where* $n \geq 1\}$.
5. $[Nat_1] = [Nat_1'] = \{M \in \mathbb{M}^1\ /\ M \rhd_\beta^* \lambda f^1.f^1$ *or* $M \rhd_\beta^* \lambda f^1.\lambda x^1.(f^1)^n y^1$ *where* $n \geq 1\}$. *(Note that $Nat_1' \notin \mathbb{U}$.)*
6. $[Nat_0'] = \{M \in \mathbb{M}^0\ /M \rhd_\beta^* \lambda f^0.f^0$ *or* $M \rhd_\beta^* \lambda f^0.\lambda y^1.f^0 y^1\}$.
7. $[(a \sqcap b) \to a] = \{M \in \mathbb{M}^0\ /M \rhd_\beta^* \lambda y^0.y^0\}$.
8. *It is not possible that* $\lambda y^0.y^0 : \langle () \vdash_1 (a \sqcap b) \to a\rangle$.
9. $\lambda y^0.y^0 : \langle () \vdash_2 (a \sqcap b) \to a\rangle$.

Proof

1. Let $y \in \mathcal{V}_2$ and $\mathcal{X} = \{M \in \mathbb{M}^0\ /\ M \rhd_\beta^* x^0 N_1 ... N_k$ where $k \geq 0$ and $x \in \mathcal{V}_1$ or $M \rhd_\beta^* y^0\}$. $\mathcal{X}$ is saturated and $\forall x \in \mathcal{V}_1, \mathcal{N}_x^0 \subseteq \mathcal{X} \subseteq \mathbb{M}^0$. Take an interpretation $\mathcal{I}$ such that $\mathcal{I}(a) = \mathcal{X}$. If $M \in [Id_0]$, then M is closed and $M \in \mathcal{X} \rightsquigarrow \mathcal{X}$. Since $y^0 \in \mathcal{X}$ and $M \diamond y^0$, then $My^0 \in \mathcal{X}$ and $My^0 \rhd_\beta^* x^0 N_1 ... N_k$ for some $x \in \mathcal{V}_1$ or $My^0 \rhd_\beta^* y^0$. Since M is closed and $x^0 \neq y^0$, by lemma 6.1, $My^0 \rhd_\beta^* y^0$. Hence, by lemma 10.4, $M \rhd_\beta^* \lambda y^0.y^0$ and, by lemma 6.(1 and 2), $M \in \mathbb{M}^0$.

   Conversely, let $M \in \mathbb{M}^0$ be closed and $M \rhd_\beta^* \lambda y^0.y^0$. Let $\mathcal{I}$ be an interpretation, $N \in \mathcal{I}(a)$ and $M \diamond N$. Since $\mathcal{I}(a)$ is saturated and $MN \rhd_\beta^* N$, $MN \in \mathcal{I}(a)$ and hence $M \in \mathcal{I}(a) \rightsquigarrow \mathcal{I}(a)$. Hence, $M \in [Id_0]$.

2. By lemma 19 (by lemma 12, $a \to a$ is good), $[Id_1'] = [ea \to ea] = [e(a \to a)] = [Id_1] = [a \to a]^+ = [Id_0]^+$. By 1., and lemma 8.1d, $[Id_0]^+ = \{M \in \mathbb{M}^1\ /\ M$ is closed and $M \rhd_\beta^* \lambda y^1.y^1\}$.

3. Let $y \in \mathcal{V}_2$, $\mathcal{X} = \{M \in \mathbb{M}^0\ /\ M \rhd_\beta^* y^0$ or $M \rhd_\beta^* x^0 N_1 ... N_k$ where $k \geq 0$ and $x \in \mathcal{V}_1\}$ and $\mathcal{Y} = \{M \in \mathbb{M}^0\ /\ M \rhd_\beta^* y^0 y^0$ or $M \rhd_\beta^* x^0 N_1 ... N_k$ or $M \rhd_\beta^* y^0(x^0 N_1 ... N_k)$ where $k \geq 0$ and $x \in \mathcal{V}_1\}$. $\mathcal{X}, \mathcal{Y}$ are saturated and $\forall x \in \mathcal{V}_1, \mathcal{N}_x^0 \subseteq \mathcal{X}, \mathcal{Y} \subseteq \mathbb{M}^0$. Let $\mathcal{I}$ be an interpretation such that $\mathcal{I}(a) = \mathcal{X}$ and $\mathcal{I}(b) = \mathcal{Y}$. If $M \in [D]$, then M is closed (hence $M \diamond y^0$) and $M \in (\mathcal{X} \cap (\mathcal{X} \rightsquigarrow \mathcal{Y})) \rightsquigarrow \mathcal{Y}$. Since $y^0 \in \mathcal{X}$ and $y^0 \in \mathcal{X} \rightsquigarrow \mathcal{Y}$, $y^0 \in \mathcal{X} \cap (\mathcal{X} \rightsquigarrow \mathcal{Y})$ and $My^0 \in \mathcal{Y}$. Since M is closed and $x^0 \neq y^0$, by lemma 6.1, $My^0 \rhd_\beta^* y^0 y^0$. Hence, by lemma 10.4, $M \rhd_\beta^* \lambda y^0.y^0 y^0$ and, by lemma 6.(1 and 2), $\mathrm{d}(M) = 0$ and $M \in \mathbb{M}^0$.

   Conversely, let $M \in \mathbb{M}^0$ be closed and $M \rhd_\beta^* \lambda y^0.y^0 y^0$. Let $\mathcal{I}$ be an interpretation and $N \in \mathcal{I}(a \sqcap (a \to b)) = \mathcal{I}(a) \cap (\mathcal{I}(a) \rightsquigarrow \mathcal{I}(b))$ (since M is closed, $M \diamond N$). Since $\mathcal{I}(b)$ is saturated, $NN \in \mathcal{I}(b)$ and $MN \rhd_\beta^* NN$, we have $MN \in \mathcal{I}(b)$ and hence $M \in \mathcal{I}(a \sqcap (a \to b)) \rightsquigarrow \mathcal{I}(b)$. Therefore, $M \in [D]$.

4. Let $f, y \in \mathcal{V}_2$ where $f \neq y$ and take $\mathcal{X} = \{M \in \mathbb{M}^0 \ / \ M \rhd_\beta^* (f^0)^n (x^0 N_1 ... N_k)$ or $M \rhd_\beta^* (f^0)^n y^0$ where $k, n \geq 0$ and $x \in \mathcal{V}_1\}$. $\mathcal{X}$ is saturated and $\forall x \in \mathcal{V}_1$, $\mathcal{N}_x^0 \subseteq \mathcal{X} \subseteq \mathbb{M}^0$. Let $\mathcal{I}$ be an interpretation such that $\mathcal{I}(a) = \mathcal{X}$. If $M \in [Nat_0]$, then M is closed and $M \in (\mathcal{X} \rightsquigarrow \mathcal{X}) \rightsquigarrow (\mathcal{X} \rightsquigarrow \mathcal{X})$. We have $f^0 \in \mathcal{X} \rightsquigarrow \mathcal{X}$, $y^0 \in \mathcal{X}$ and $\diamond\{M, f^0, y^0\}$, then $M f^0 y^0 \in \mathcal{X}$ and $M f^0\, y^0 \rhd_\beta^* (f^0)^n (x^0 N_1 ... N_k)$ or $M f^0 y^0 \rhd_\beta^* (f^0)^n y^0$ where $n \geq 0$. Since M is closed and $\{x^0\} \cap \{y^0, f^0\} = \emptyset$, by lemma 6.1, $M f^0 y^0 \rhd_\beta^* (f^0)^n y^0$ where $n \geq 1$. Hence, by lemma 10.4, $M \rhd_\beta^* \lambda f^0.f^0$ or $M \rhd_\beta^* \lambda f^0.\lambda y^0.(f^0)^n y^0$ where $n \geq 1$. Moreover, by lemma 6.(1 and 2), $\mathrm{d}(M) = 0$ and $M \in \mathbb{M}^0$.

   Conversely, let $M \in \mathbb{M}^0$ be closed and $M \rhd_\beta^* \lambda f^0.f^0$ or $M \rhd_\beta^* \lambda f^0.\lambda y^0.(f^0)^n y^0$ where $n \geq 1$. Let $\mathcal{I}$ be an interpretation, $N \in \mathcal{I}(a \to a) = \mathcal{I}(a) \rightsquigarrow \mathcal{I}(a)$, $N' \in \mathcal{I}(a)$ and $N \diamond N'$. We show, by induction on $m \geq 0$, that $(N)^m N' \in \mathcal{I}(a)$. Since $M N N' \rhd_\beta^* (N)^m N'$ where $m \geq 0$ and $(N)^m N' \in \mathcal{I}(a)$ which is saturated, then $M N N' \in \mathcal{I}(a)$. Hence, $M \in (\mathcal{I}(a) \rightsquigarrow \mathcal{I}(a)) \to (\mathcal{I}(a) \rightsquigarrow \mathcal{I}(a))$ and $M \in [Nat_0]$.

5. By lemma 19, $[Nat_1] = [eNat_0] = [Nat_0]^+$. Let $\mathcal{I}$ be an interpretation. Since $ea \to ea$ and $e(a \to a) \to e(a \to a)$ are good (by lemma 12), then, by lemmas 15.5 and 19.3, $\mathcal{I}(e(a \to a) \to (ea \to ea)) = \mathcal{I}((a \to a) \to (a \to a))^+$ and hence $[Nat_1'] = [Nat_0]^+$. By 4., $[Nat_1] = [Nat_1'] = [Nat_0]^+ = \{M \in \mathbb{M}^1$ $/ \ M$ is closed and $M \rhd_\beta^* \lambda f^1.f^1$ or $M \rhd_\beta^* \lambda f^1.\lambda y^1.(f^1)^n y^1$ where $n \geq 1\}$.

6. Let $f, y \in \mathcal{V}_2$ where $f \neq y$ and take $\mathcal{X} = \{M \in \mathbb{M}^0 \ / \ M \rhd_\beta^* x^0 P_1 ... P_k$ or $M \rhd_\beta^* f^0\ (x^1 Q_1 ... Q_l)$ or $M \rhd_\beta^* y^0$ or $M \rhd_\beta^* f^0 y^1$ where $k, l \geq 0$, $x \in \mathcal{V}_1$ and $\mathrm{d}(Q_i) \geq 1\}$. $\mathcal{X}$ is saturated and $\forall x \in \mathcal{V}_1$, $\mathcal{N}_x^0 \subseteq \mathcal{X} \subseteq \mathbb{M}^0$. Let $\mathcal{I}$ be an interpretation such that $\mathcal{I}(a) = \mathcal{X}$. If $M \in [Nat_0']$, then M is closed and $M \in (\mathcal{X}^+ \rightsquigarrow \mathcal{X}) \rightsquigarrow (\mathcal{X}^+ \rightsquigarrow \mathcal{X})$. Let $N \in \mathcal{X}^+$ such that $f^0 \diamond N$. We have $N \rhd_\beta^* x^1 P_1^+ ... P_k^+$ or $N \rhd_\beta^* y^1$, then $f^0 N \rhd_\beta^* f^0(x^1 P_1^+ ... P_k^+) \in \mathcal{X}$ or $N \rhd_\beta^* f^0 y^1 \in \mathcal{X}$, thus $f^0 \in \mathcal{X}^+ \rightsquigarrow \mathcal{X}$. We have $f^0 \in \mathcal{X}^+ \rightsquigarrow \mathcal{X}$, $y^1 \in \mathcal{X}^+$ and $\diamond\{M, f^0, y^1\}$, then $M f^0 y^1 \in \mathcal{X}$. Since M is closed and $\{x^0, x^1\} \cap \{y^1, f^0\} = \emptyset$, by lemma 6.1, $M f^0 y^1 \rhd_\beta^* f^0 y^1$. Hence, by lemma 10.4, $M \rhd_\beta^* \lambda f^0.f^0$ or $M \rhd_\beta^* \lambda f^0.\lambda y^1.f^0 y^1$. Moreover, by lemma 6.(1 and 2), $\mathrm{d}(M) = 0$ and $M \in \mathbb{M}^0$.

   Conversely, let $M \in \mathbb{M}^0$ be closed and $M \rhd_\beta^* \lambda f^0.f^0$ or $M \rhd_\beta^* \lambda f^0.\lambda y^1.f^0 y^1$. Let $\mathcal{I}$ be an interpretation, $N \in \mathcal{I}(ea \to a) = \mathcal{I}(a)^+ \rightsquigarrow \mathcal{I}(a)$ and $N' \in \mathcal{I}(a)^+$ where $\diamond\{M, N, N'\}$. Since $M N N' \rhd_\beta^* N N'$, $N N' \in \mathcal{I}(a)$ and $\mathcal{I}(a)$ is saturated, then $M N N' \in \mathcal{I}(a)$. Hence, $M \in (\mathcal{I}(a)^+ \rightsquigarrow \mathcal{I}(a)) \to (\mathcal{I}(a)^+ \rightsquigarrow \mathcal{I}(a))$ and $M \in [Nat_0']$.

7. Let $y \in \mathcal{V}_2$ and take $\mathcal{X} = \{M \in \mathbb{M}^0 \ / \ M \rhd_\beta^* y^0$ or $M \rhd_\beta^* x^0 N_1 ... N_k$ where $k \geq 0$ and $x \in \mathcal{V}_1\}$. $\mathcal{X}$ is saturated and $\forall x \in \mathcal{V}_1$, $\mathcal{N}_x^0 \subseteq \mathcal{X} \subseteq \mathbb{M}^0$. Let $\mathcal{I}$ be an interpretation such that $\mathcal{I}(a) = \mathcal{I}(b) = \mathcal{X}$. If $M \in [(a \sqcap b) \to a]$, then M is closed and $M \in \mathcal{X} \rightsquigarrow \mathcal{X}$. Since $M\ y^0 \in \mathcal{X}$ (as $y^0 \in \mathcal{X}$ and $M \diamond y^0$) and M is closed and $x^0 \neq y^0$, by lemma 6.1, $M\ y^0 \rhd_\beta^* y^0$. Hence, by lemma 10.4, $M \rhd_\beta^* \lambda y^0.y^0$. By lemma 6.(1 and 2), $\mathrm{d}(M) = \mathrm{d}(\lambda y^0.y^0) = 0$ and $M \in \mathbb{M}^0$.

   Conversely, let $M \in \mathbb{M}^0$ be closed and $M \rhd_\beta^* \lambda y^0.y^0$. Let $\mathcal{I}$ be an interpretation and $N \in \mathcal{I}(a \sqcap b)$ (hence $M \diamond N$). Since $\mathcal{I}(a)$ is saturated, $N \in \mathcal{I}(a)$ and $MN \rhd_\beta^* N$, then $MN \in \mathcal{I}(a)$ and hence $M \in \mathcal{I}(a \sqcap b) \rightsquigarrow \mathcal{I}(a)$. Hence, $M \in [(a \sqcap b) \to a]$.

8. If $\lambda y^0.y^0 : \langle () \vdash_1 (a \sqcap b) \to a \rangle$, then, by Lemma 25, $y^0 : \langle (y^0 : a \sqcap b) \vdash_1 a \rangle$ and again, by Lemma 25, $y^0 : a = y^0 : a \sqcap b$. Hence, $a = a \sqcap b$. Absurd.

9. Easy.

$\square$

Remark 42 (Failure of completeness for $\vdash_1$) *Items 7. and 8. of lemma 41 show that we can not have a completeness result (a converse of theorem 39) for $\vdash_1$. To type the term $\lambda y^0.y^0$ by the type $(a \sqcap b) \to a$, we need an elimination rule for $\sqcap$ which we have in $\vdash_2$. However, we will see that we have completeness for $\vdash_2$ only if we are restricted to the use of one single expansion variable.*

6 Completeness of $\vdash_2$ with one expansion variable

Recall remark 42 where we said that $\lambda y^0.y^0 : \langle () \vdash_2 (a \sqcap b) \to a\rangle$ and that we even have completeness for one single expansion variable in the new type system. In this section, we will establish this completeness theorem for one expansion variable. First, we give an example (see lemma 43) which shows why completeness does not work in the presence of more than one expansion variable.

Lemma 43 *Let $Nat''_0 = (e_1 a \to a) \to (e_2 a \to a)$ where $a \in \mathcal{A}$, $e_1, e_2 \in \mathcal{E}$ and $e_1 \neq e_2$. We have:*

1. *$\lambda f^0.f^0 \in [Nat''_0]$.*
2. *It is not possible that $\lambda f^0.f^0 : \langle () \vdash_2 Nat''_0\rangle$.*

Proof 1. For every interpretation $\mathcal{I}$, $\mathcal{I}(e_1 a \to a) = \mathcal{I}(e_2 a \to a) = \mathcal{I}(a)^+ \rightsquigarrow \mathcal{I}(a)$.
2. If $\lambda f^0.f^0 : \langle () \vdash_2 Nat''_0\rangle$, by lemma 26.2 and 26.1, $f^0 : \langle f^0 : e_1 a \to a \vdash_2 e_2 a \to a\rangle$ and $e_1 a \to a \sqsubseteq e_2 a \to a$. Thus, by lemma 21.7, $e_2 a \sqsubseteq e_1 a$. Again, by lemma 21.5, $e_1 a = e_2 U$ where $a \sqsubseteq U$. This is impossible since $e_1 \neq e_2$. $\square$

Hence we have $\lambda f^0.f^0 \in [Nat''_0]$ but $\lambda f^0.f^0$ is not typable by Nat''_0 and we do not have completeness in the presence of more than one expansion variable. The problem comes from the fact that for the realizability semantics that we considered, we identify all expansion variables. In order to give a completeness theorem we will in what follows restrict our system to only one expansion variable. In the rest of this section, we assume that the set $\mathcal{E}$ contains only one expansion variable e_c.

The need of one single expansion variable is clear in part 2) of the next lemma which would fail if we use more than one single expansion variable. For example, if $e_1 \neq e_2$ then $e_1(e_2 a)^- = e_1 a \neq e_2 a$. The next lemma is crucial for the rest of this section and hence, having a single expansion variable is also crucial.

Lemma 44 *Let $U, V \in \mathbb{U}$ and $d(U) = d(V) > 0$.*

1. *$e_c U^- = U$.*
2. *If $U^- = V^-$, then $U = V$.*

Proof 1. By induction on U. 2. If $U^- = V^-$ then $e_c U^- = e_c V^-$ and by 1, $U = V$. $\square$

In the next definition and lemma, we divide the set $\{y^n \ / \ y \in \mathcal{V}_2\}$ disjointly amongst the types of order n.

Definition 45 *Let $U \in \mathbb{U}$. We inductively define sets of variables $\mathbb{V}_U$ by:*

- *If $d(U) = 0$, then:*
  - *$\mathbb{V}_U$ is an infinite set of variables of degree 0.*
  - *If $y^0 \in \mathbb{V}_U$, then $y \in \mathcal{V}_2$.*
  - *If $U \neq V$ and $d(U) = d(V) = 0$, then $\mathbb{V}_U \cap \mathbb{V}_V = \emptyset$.*

- *If $d(U) = n+1$, then we put $\mathbb{V}_U = \{y^{n+1} \ / \ y^n \in \mathbb{V}_{U^-}\}$.*

Lemma 46 *1. If $d(U) = n$, then $\mathbb{V}_U$ is an infinite set of variables of degree n and if $y^n \in \mathbb{V}_U$, then $y \in \mathcal{V}_2$.*

2. If $U \neq V$ and $d(U) = d(V) = n$, then $\mathbb{V}_U \cap \mathbb{V}_V = \emptyset$.

3. If $y^n \in \mathbb{V}_U$, then $y^{n+1} \in \mathbb{V}_{e_c U}$.

4. If $y^{n+1} \in \mathbb{V}_U$, then $y^n \in \mathbb{V}_{U^-}$.

Proof 1. and 2. By induction on n and using lemma 44. 3. Because $(e_c U)^- = U$. 4. By definition. □

Our partition of the set $\mathcal{V}_2$ as above will enable us to define useful infinite sets which will contain type environments that will play a crucial role in one particular type interpretation. These infinite sets and type environments are given in the next definition.

Definition 47 *1. Let $n \in \mathbb{N}$. We let $\mathbb{G}^n = \{(y^n : U) \ / \ U \in \mathbb{U},\ d(U) = n$ and $y^n \in \mathbb{V}_U\}$ and $\mathbb{H}^n = \bigcup_{m \geq n} \mathbb{G}^m$. Note that $\mathbb{G}^n$ and $\mathbb{H}^n$ are not type environments because they are infinite sets.*

2. Let $n \in \mathbb{N}$, $M \in \mathcal{M}$ and $U \in \mathbb{U}$, we write $M : \langle \mathbb{H}^n \vdash_2 U \rangle$ iff there is a type environment $\Gamma \subset \mathbb{H}^n$ where $M : \langle \Gamma \vdash_2 U \rangle$

Lemma 48 *1. If $\Gamma \subset \mathbb{H}^n$, then $e\Gamma \subset \mathbb{H}^{n+1}$.*

2. If $\Gamma \subset \mathbb{H}^{n+1}$, then $\Gamma^- \subset \mathbb{H}^n$.

3. If $\Gamma_1 \subset \mathbb{H}^n$, $\Gamma_2 \subset \mathbb{H}^m$ and $m \geq n$, then $\Gamma_1 \sqcap \Gamma_2 \subset \mathbb{H}^n$.

Proof 1. resp. 2. By lemma 46.3 resp. 46.4. 3. First note that $\mathbb{H}^m \subseteq \mathbb{H}^n$. Let $(x^p : U_1 \sqcap U_2) \in \Gamma_1 \sqcap \Gamma_2$ where $(x^p : U_1) \in \Gamma_1 \subset \mathbb{H}^n$ and $(x^p : U_2) \in \Gamma_2 \subset \mathbb{H}^m \subseteq \mathbb{H}^n$, then $\mathrm{d}(U_1) = \mathrm{d}(U_2) = p$ and $x^p \in \mathbb{V}_{U_1} \cap \mathbb{V}_{U_2}$. Hence, by lemma 46.2, $U_1 = U_2$ and $\Gamma_1 \sqcap \Gamma_2 = \Gamma_1 \cup \Gamma_2 \subset \mathbb{H}^n$. □

Now, for every n, we define the set of the good terms of order n which contain some free variable x^i where $x \in \mathcal{V}_1$ and $i \geq n$.

Definition 49 *For every $n \in \mathbb{N}$, let $\mathcal{V}^n = \{M \in \mathbb{M}^n \ / \ x^i \in FV(M)$ where $x \in \mathcal{V}_1$ and $i \geq n\}$. It is easy to see that, for every $n \in \mathbb{N}$ and for every $x \in \mathcal{V}_1$, $\mathcal{N}_x^n \subseteq \mathcal{V}^n$.*

Lemma 50 *1. $(\mathcal{V}^n)^+ = \mathcal{V}^{n+1}$.*

2. If $y \in \mathcal{V}_2$ and $(M\ y^m) \in \mathcal{V}^n$, then $M \in \mathcal{V}^n$.

3. If $M \in \mathcal{V}^n$, $M \diamond N$, $N \in \mathbb{M}$ and $d(N) = m \geq n$, then $MN \in \mathcal{V}^n$.

4. If $d(M) = n$, $m \geq n$, $M \diamond N$, $M \in \mathbb{M}$ and $N \in \mathcal{V}^m$, then $MN \in \mathcal{V}^n$.

Proof Easy. □

Finally, the crucial interpretation $\mathbb{I}$ for the proof of completeness is given as follows:

Definition 51 *Let $\mathbb{I}$ be the interpretation defined by: for all type variables a,* $\mathbb{I}(a) = \mathcal{V}^0 \cup \{M \in \mathcal{M}^0 \ / \ M : \langle \mathbb{H}^0 \vdash_2 a\rangle\}$.

The next lemma shows that $\mathbb{I}$ is indeed an interpretation and moreover, the interpretation of a type of order n contains the good terms of order n which are typable in these special environments which are parts of the infinite sets of definition 47.

Lemma 52 *1. $\mathbb{I}$ is an interpretation. I.e., $\forall a \in \mathcal{A}$, $\mathbb{I}(a)$ is saturated and $\forall x \in \mathcal{V}_1$, $\mathcal{N}^0_x \subseteq \mathbb{I}(a) \subseteq \mathbb{M}^0$.*

2. If $U \in \mathbb{U}$ is good and $d(U) = n$, then
$\mathbb{I}(U) = \mathcal{V}^n \cup \{M \in \mathbb{M}^n \ / \ M : \langle \mathbb{H}^n \vdash_2 U\rangle\}$.

Proof 1. First we show that $\mathbb{I}(a)$ is saturated. Let $M \rhd^*_\beta N$ and $N \in \mathbb{I}(a)$.

- If $N \in \mathcal{V}^0$ then $N \in \mathbb{M}^0$ and $\exists x^i$ such that $x \in \mathcal{V}_1$, $i \geq 0$ and $x^i \in FV(N)$. By lemma 15.6, $\mathbb{M}^0$ is saturated and so, $M \in \mathbb{M}^0$. By lemma 6.1, $FV(M) = FV(N)$ and so, $x^i \in FV(M)$. Hence, $M \in \mathcal{V}^0$
- If $N \in \{M \in \mathcal{M}^0 \ / \ M : \langle \mathbb{H}^0 \vdash_2 a\rangle\}$ then $\exists \Gamma \subset \mathbb{H}^0$, such that $N : \langle \Gamma \vdash_2 a\rangle$. By subject expansion corollary 36, $M : \langle \Gamma \vdash_2 a\rangle$ and by lemma 6.1, $\mathrm{d}(M) = \mathrm{d}(N)$. Hence, $M \in \{M \in \mathcal{M}^0 \ / \ M : \langle \mathbb{H}^0 \vdash_2 a\rangle\}$.

Now we show that $\forall x \in \mathcal{V}_1$, $\mathcal{N}^0_x \subseteq \mathbb{I}(a) \subseteq \mathbb{M}^0$.

- Let $x \in \mathcal{V}_1$ and $M \in \mathcal{N}^0_x$. Hence, $M = x^0 N_1 \dots N_k \in \mathbb{M}^0$, and $x^0 \in FV(M)$. Thus, $M \in \mathcal{V}^0$.
- Let $M \in \mathbb{I}(a)$. If $M \in \mathcal{V}^0$, then $M \in \mathbb{M}^0$. Else, $\exists \Gamma \subset \mathbb{H}^0$ where $M : \langle \Gamma \vdash_2 a\rangle$. Since by lemma 23, M is good and $\mathrm{d}(M) = \mathrm{d}(a) = 0$, $M \in \mathbb{M}^0$.

2. By induction on U good.

- $U = a$: By definition of $\mathbb{I}$ and by 1.
- $U = e_c V$: $\mathrm{d}(V) = n-1$ and, by lemma 12, V is good. By IH and lemma 50.1, $\mathbb{I}(e_c V) = (\mathbb{I}(V))^+ = (\mathcal{V}^{n-1} \cup \{M \in \mathbb{M}^{n-1} \ / \ M : \langle \mathbb{H}^{n-1} \vdash_2 V\rangle\})^+ = \mathcal{V}^n \cup (\{M \in \mathbb{M}^{n-1} \ / \ M : \langle \mathbb{H}^{n-1} \vdash_2 V\rangle\})^+$.
  - If $M \in \mathbb{M}^{n-1}$ and $M : \langle \mathbb{H}^{n-1} \vdash_2 V\rangle$, then $M : \langle \Gamma \vdash_2 V\rangle$ where $\Gamma \subset \mathbb{H}^{n-1}$. By *exp* and lemma 48.1, $M^+ : \langle e_c\Gamma \vdash_2 e_c V\rangle$ and $e_c\Gamma \subset \mathbb{H}^n$. Thus by lemma 23.(3 and 1), $M^+ \in \mathbb{M}^n$ and $M^+ : \langle \mathbb{H}^n \vdash_2 e_c V\rangle$.
  - If $M \in \mathbb{M}^n$ and $M : \langle \mathbb{H}^n \vdash_2 e_c V\rangle$, then $M : \langle \Gamma \vdash_2 e_c V\rangle$ where $\Gamma \subset \mathbb{H}^n$. By lemmas 30.5, and 48.2, $M^- : \langle \Gamma^- \vdash_2 V\rangle$ and $\Gamma^- \subset \mathbb{H}^{n-1}$. Thus, by lemma 8.(1b and 1d), $M = (M^-)^+$ and $M^- \in \mathbb{M}^{n-1}$. Hence, $M^- \in \{M \in \mathbb{M}^{n-1} \ / \ M : \langle \mathbb{H}^{n-1} \vdash_2 V\rangle\}$.

  Hence $(\{M \in \mathbb{M}^{n-1} \ / \ M : \langle \mathbb{H}^{n-1} \vdash_2 V\rangle\})^+ = \{M \in \mathbb{M}^n \ / \ M : \langle \mathbb{H}^n \vdash_2 U\rangle\}$ and $\mathbb{I}(U) = \mathcal{V}^n \cup \{M \in \mathbb{M}^n \ / \ M : \langle \mathbb{H}^n \vdash_2 U\rangle\}$.
- $U = U_1 \sqcap U_2$: By lemma 12, U_1, U_2 are good and $\mathrm{d}(U_1) = \mathrm{d}(U_2) = n$. By IH, $\mathbb{I}(U_1 \sqcap U_2) = \mathbb{I}(U_1) \cap \mathbb{I}(U_2) =$
  $(\mathcal{V}^n \cup \{M \in \mathbb{M}^n \ / \ M : \langle \mathbb{H}^n \vdash_2 U_1\rangle\}) \cap (\mathcal{V}^n \cup \{M \in \mathbb{M}^n \ / \ M : \langle \mathbb{H}^n \vdash_2 U_2\rangle\}) =$
  $\mathcal{V}^n \cup (\{M \in \mathbb{M}^n \ / \ M : \langle \mathbb{H}^n \vdash_2 U_1\rangle\} \cap \{M \in \mathbb{M}^n \ / \ M : \langle \mathbb{H}^n \vdash_2 U_2\rangle\})$.
  - If $M \in \mathbb{M}^n$, $M : \langle \mathbb{H}^n \vdash_2 U_1\rangle$ and $M : \langle \mathbb{H}^n \vdash_2 U_2\rangle$, then $M : \langle \Gamma_1 \vdash_2 U_1\rangle$ and $M : \langle \Gamma_2 \vdash_2 U_2\rangle$ where $\Gamma_1, \Gamma_2 \subset \mathbb{H}^n$. By remark 24, $M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_2 U_1 \sqcap U_2\rangle$. Since by lemma 48.3, $\Gamma_1 \sqcap \Gamma_2 \subset \mathbb{H}^n$, $M : \langle \mathbb{H}^n \vdash_2 U_1 \sqcap U_2\rangle$.

- If $M \in \mathbb{M}^n$ and $M : \langle \mathbb{H}^n \vdash_2 U_1 \sqcap U_2 \rangle$, then $M : \langle \Gamma \vdash_2 U_1 \sqcap U_2 \rangle$ where $\Gamma \subset \mathbb{H}^n$. By $\sqsubseteq$, $M : \langle \Gamma \vdash_2 U_1 \rangle$ and $M : \langle \Gamma \vdash_2 U_2 \rangle$. Hence, $M : \langle \mathbb{H}^n \vdash_2 U_1 \rangle$ and $M : \langle \mathbb{H}^n \vdash_2 U_2 \rangle$.

  We deduce that $\mathbb{I}(U_1 \sqcap U_2) = \mathcal{V}^n \cup \{M \in \mathbb{M}^n \ / \ M : \langle \mathbb{H}^n \vdash_2 U_1 \sqcap U_2 \rangle\}$.

- $U = V \to T$: By lemmas 12.2, V, T are good and let $m = \mathrm{d}(V) \geq \mathrm{d}(T) = 0$. By IH, $\mathbb{I}(V) = \mathcal{V}^m \cup \{M \in \mathbb{M}^m \ / \ M : \langle \mathbb{H}^m \vdash_2 V \rangle\}$ and $\mathbb{I}(T) = \mathcal{V}^0 \cup \{M \in \mathbb{M}^0 \ / \ M : \langle \mathbb{H}^0 \vdash_2 T \rangle\}$. Note that $\mathbb{I}(V \to T) = \mathbb{I}(V) \rightsquigarrow \mathbb{I}(T)$.

  - Let $M \in \mathbb{I}(V) \rightsquigarrow \mathbb{I}(T)$ and by lemma 46.1, let $y^m \in \mathbb{V}_V$ such that $y \in \mathcal{V}_2$, and $\forall n, y^n \notin FV(M)$. Then $y^m \diamond M$. By remark 24, $y^m : \langle (y^m : V) \vdash_2 V \rangle$. Hence $y^m : \langle \mathbb{H}^m \vdash_2 V \rangle$ and so $y^m \in \mathbb{I}(V)$ and $My^m \in \mathbb{I}(T)$.
    * If $My^m \in \mathcal{V}^0$, then since $y \in \mathcal{V}_2$, by lemma 50.2, $M \in \mathcal{V}^0$.
    * If $My^m \in \{M \in \mathbb{M}^0 \ / \ M : \langle \mathbb{H}^0 \vdash_2 T \rangle\}$ then $My^m \in \mathbb{M}^0$ and $My^m : \langle \mathbb{H}^0 \vdash_2 T \rangle$. So $My^m : \langle \Gamma \vdash_2 T \rangle$ where $\Gamma \subset \mathbb{H}^0$. Since $y^m \in FV(My^m)$ and since by lemma 22, $dom(\Gamma) = FV(My^m)$, $\Gamma = \Gamma', (y^m : V')$ where by lemma 23.1, $\mathrm{d}(V') = m$. Since $(y^m : V') \in \mathbb{H}^0$, $\mathrm{d}(V') = m$ and $y^m \in \mathbb{V}_{V'}$, by lemma 46.2, $V = V'$. So $My^m : \langle \Gamma', (y^m : V) \vdash_2 T \rangle$ and by lemma 26.1, $M : \langle \Gamma' \vdash_2 V \to T \rangle$ and by lemma 23.(1 and 3), $M \in \mathbb{M}$ and $\mathrm{d}(M) = 0$. Since $\Gamma' \subset \mathbb{H}^0$, $M : \langle \mathbb{H}^0 \vdash_2 V \to T \rangle$. And so, $M \in \{M \in \mathbb{M}^0 \ / \ M : \langle \mathbb{H}^0 \vdash_2 V \to T \rangle\}$.
  - Let $M \in \mathcal{V}^0 \cup \{M \in \mathbb{M}^0 \ / \ M : \langle \mathbb{H}^0 \vdash_2 V \to T \rangle\}$ and $N \in \mathbb{I}(V) = \mathcal{V}^m \cup \{M \in \mathbb{M}^m \ / \ M : \langle \mathbb{H}^m \vdash_2 V \rangle\}$ such that $M \diamond N$. Then, $\mathrm{d}(N) = m$.
    * Case $M \in \mathcal{V}^0$. Since $N \in \mathbb{M}$, by lemma 50.3, $MN \in \mathcal{V}^0 \subseteq \mathbb{I}(T)$.
    * Case $M \in \{M \in \mathbb{M}^0 \ / \ M : \langle \mathbb{H}^0 \vdash_2 V \to T \rangle\}$, so $M \in \mathbb{M}^0$.
      · If $N \in \mathcal{V}^m$, then, by lemma 50.4, $MN \in \mathcal{V}^0 \subseteq \mathbb{I}(T)$.
      · If $N \in \{M \in \mathbb{M}^m \ / \ M : \langle \mathbb{H}^m \vdash_2 V \rangle\}$, so $M : \langle \Gamma_1 \vdash_2 V \to T \rangle$ and $N : \langle \Gamma_2 \vdash_2 V \rangle$ where $\Gamma_1 \subset \mathbb{H}^0$ and $\Gamma_2 \subset \mathbb{H}^m$. Since $M \diamond N$ then by lemma 22.4, $\Gamma_1 \diamond \Gamma_2$. So by $\to_e$, $MN : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_2 T \rangle$. By lemma 48.3, $\Gamma_1 \sqcap \Gamma_2 \subset \mathbb{H}^0$. Therefore $MN : \langle \mathbb{H}^0 \vdash_2 T \rangle$. By lemma 23, $MN \in \mathbb{M}^0$. Hence, $MN \in \{M \in \mathbb{M}^0 \ / \ M : \langle \mathbb{H}^0 \vdash_2 T \rangle\} \subseteq \mathbb{I}(T)$.

    Hence, $M \in \mathbb{I}(V \to T)$.

  We deduce that $\mathbb{I}(V \to T) = \mathcal{V}^0 \cup \{M \in \mathbb{M}^0 \ / \ M : \langle \mathbb{H}^0 \vdash_2 V \to T \rangle\}$.

$\square$

Now we use the special $\mathbb{I}$ to show completeness.

Theorem 53 (Completeness) *Let $U \in \mathbb{U}$ be good such that $d(U) = n$.*

1. *$[U] = \{M \in \mathbb{M}^n \ / \ M : \langle () \vdash_2 U \rangle\}$.*
2. *$[U]$ is stable by reduction: i.e., if $M \in [U]$ and $M \rhd_\beta^* N$, then $N \in [U]$.*
3. *$[U]$ is stable by expansion: i.e., if $N \in [U]$ and $M \rhd_\beta^* N$, then $M \in [U]$.*

Proof Recall: $[U] = \{M \in \mathcal{M} \ / \ M \textit{ is closed and } M \in \bigcap_{\mathcal{I} \ \ interpretation} \mathcal{I}(U)\}$.

1. Let $M \in [U]$. Then M is a closed term and $M \in \mathbb{I}(U)$. Hence, by lemma 52, $M \in \mathcal{V}^n \cup \{M \in \mathbb{M}^n \ / \ M : \langle \mathbb{H}^n \vdash_2 U \rangle\}$. Since M is closed, $M \notin \mathcal{V}^n$. Hence, $M \in \{M \in \mathbb{M}^n \ / \ M : \langle \mathbb{H}^n \vdash_2 U \rangle\}$ and so, $M : \langle \Gamma \vdash_2 U \rangle$ where $\Gamma \subset \mathbb{H}^n$. Since M is closed, by lemma 22.1, $\Gamma = ()$ and $M : \langle () \vdash_2 U \rangle$.

   Conversely, let $M \in \mathbb{M}^n$ where $M : \langle () \vdash_2 U \rangle$. By lemma 22.1, M is closed. Let $\mathcal{I}$ be an interpretation. By soundness lemma 38, $M \in \mathcal{I}(U)$. Thus $M \in [U]$.

2. Let $M \in [U]$ such that $M \rhd_\beta^* N$. By 1, $M \in \mathbb{M}^n$ and $M : \langle () \vdash_2 U \rangle$. By subject reduction corollary 33, $N : \langle () \vdash_2 U \rangle$. By lemma 6.1, $\mathrm{d}(N) = \mathrm{d}(M) = n$. By lemma 23.3, $N \in \mathbb{M}$. Hence, by 1, $N \in [U]$.
3. Let $N \in [U]$ such that $M \rhd_\beta^* N$. By 1, $N \in \mathbb{M}^n$ and $N : \langle () \vdash_2 U \rangle$. By subject expansion corollary 36, $M : \langle () \vdash_2 U \rangle$. By lemma 6.1, $\mathrm{d}(N) = \mathrm{d}(M) = n$. By lemma 23.3, $M \in \mathbb{M}$. Hence, by 1, $M \in [U]$.

□

7 Conclusion and future work

In this paper, we studied the $\lambda I^{\mathbb{N}}$-calculus, an indexed version of the λI-calculus. This indexed version was typed using first a basic intersection type system with expansion variables but without an intersection elimination rule, and then using an intersection type system with expansion variables and an elimination rule.

We gave a realisability semantics for both type systems showing that the first type system is not complete in the sense that there are types whose semantic meaning is not the set of $\lambda I^{\mathbb{N}}$-terms having this type. In particular, we showed that $\lambda y^0.y^0$ is in the semantic meaning of $(a \sqcap b) \to a$ but it is not possible to give $\lambda y^0.y^0$ the type $(a \sqcap b) \to a$. The main reason for the failure of completeness in the first system is associated with the failure of the subject reduction property for this first system. Hence, we moved to the second system which we show to have the desirable properties of subject reduction and expansion and strong normalisation. However, for this second system, we show again that completeness fails if we use more than one expansion variable but that completeness succeeds if we restrict the system to one single expansion variable.

Since we show in the appendices that each of these type systems, when restricted to the normal λI-calculus represents a well known intersection type system with expansion variables, our study can be said to be the first denotational semantics study of intersection type systems with expansion variables (using realizability or any other approach) and outlines the difficulties of doing so. Although we have in this paper limited the study to the λI-calculus, future work will include extending this work to the full λ-calculus and with an ω-type rule as well.

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A Introducing eta reduction

Now we define the eta reduction relation on the $\lambda I^{\mathbb{N}}$-calculus.

Definition 54 *1. The reduction relation $\rhd_\eta$ on $\mathcal{M}$ is defined as the least compatible relation closed under the rule: $\lambda x^n.Mx^n \rhd_\eta M$ if $x^n \notin FV(M)$ and $d(M) \leq n$.*

2. We define $\rhd_{\beta\eta} = \rhd_\beta \cup \rhd_\eta$.

3. For $r \in \{\eta, \beta\eta\}$, we denote by $\rhd_r^*$ the reflexive and transitive closure of $\rhd_r$. We also denote by $\simeq_r$ the equivalence relation induced by $\rhd_r^*$.

The next lemma shows that these new reduction relations are well defined on the $\lambda I^{\mathbb{N}}$-calculus.

Lemma 55 *Let $r \in \{\eta, \beta\eta\}$. $\rhd_r$ is a well defined relation on $\mathcal{M}$. I.e., if $M \in \mathcal{M}$ and $M \rhd_r N$ then $N \in \mathcal{M}$. Hence, $\rhd_r^*$ is a also well defined relation on $\mathcal{M}$.*

Proof The cases $r = \eta$ are by induction on $M \rhd_r N$. We only treat the basic case. Assume $\lambda x^n.Nx^n \rhd_\eta N$ where $x^n \notin FV(N)$ and $\mathrm{d}(N) \leq n$. Since N is a subterm of $\lambda x^n.Nx^n$, then $N \in \mathcal{M}$.

Now, $\rhd_{\beta\eta}$ is a relation since it is the union of two relations. Finally, we show by induction on $M \rhd_r^* N$ that if $M \in \mathcal{M}$ and $M \rhd_r^* N$ then $N \in \mathcal{M}$. □

The next lemma shows that the new reduction relations preserve the free variables, degrees and goodness of terms.

Lemma 56 *Let $M, N \in \mathcal{M}$ and $r \in \{\eta, \beta\eta\}$. Assume $M \rhd_r^* N$. We have:*

1. *$FV(M) = FV(N)$ and $d(M) = d(N)$.*
2. *M is good iff N is good.*

Proof 1. By induction on the derivation of $M \rhd_r^* N$. We only treat the following:

- Assume $\lambda x^n.Nx^n \rhd_\eta N$ where $x^n \not\in FV(N)$ and $\mathrm{d}(N) \leq n$. Obviously, $FV(\lambda x^n.Nx^n) = FV(N)$ and $\mathrm{d}(\lambda x^n.Nx^n) = \min(\mathrm{d}(N), n) = \mathrm{d}(N)$.

2. By induction on the length of the derivation $M \rhd_r^* N$.

- If the length of the derivation is 0, nothing to prove.
- Case $M \rhd_\eta N$. We do the proof by induction on the derivation of $M \rhd_\eta N$. Since the compatibility cases are similar to those of the proof of lemma 6 (when $M \rhd_\beta N$), we only do the case $M = \lambda x^n.Nx^n \rhd_\beta N$ with $\mathrm{d}(N) \leq n$ and $x^n \not\in FV(N)$. Since $M \in \mathcal{M}$, then $Nx^n \in \mathcal{M}$, and so $N \diamond x^n$.
  - If N is good then since by definition x^n is good, and since $\mathrm{d}(N) \leq n = deg(x^n)$ then by defintion Nx^n is good. Now, since $x^n \in FV(Nx^n)$, $\lambda x^n.Nx^n$ is good by definition.
  - If M is good then by lemma 2 twice, N is good.
- Case $M \rhd_{\beta\eta} N$. Then, either $M \rhd_\beta N$ or $M \rhd_\eta N$ and we use either lemma 6 or the case above.
- Case $M \rhd_r N_1 \rhd_r^* N$ use IH.

$\square$

Lemma 57 *Let $r \in \{\eta, \beta\eta\}$, $\succ \in \{\rhd, \rhd^*\}$, $p \geq 0$ and $M, N \in \mathcal{M}$.*

1. *If $M \succ_r N$, then $M^+ \succ_r N^+$.*
2. *If $d(M) > 0$ and $M \succ_r N$, then $M^- \succ_r N^-$.*
3. *If $M \succ_r N^+$, then $M^- \succ_r N$.*
4. *If $M^+ \succ_r N$, then $M \succ_r N^-$.*
5. *Let $P \in \mathcal{M}$. If $M \succ_r N$, $P \succ_r Q$ and $M \diamond P$, then $N \diamond Q$.*
6. *If $M \succ_r N$, $M \diamond P$ and $d(P) = n$, then $M[x^n := P] \succ_r N[x^n := P]$.*
7. *If $N \succ_r P$ and $M \diamond N$, then $M[x^n := N] \rhd_r^* M[x^n := P]$.*
8. *If $M \rhd_r^* N$, $P \rhd_r^* P'$, $M \diamond P$ and $d(P) = n$, then $M[x^n := P] \rhd_r^* N[x^n := P']$.*

Proof

1. The case $r \in \{\eta\}$ and $\succ = \rhd$ is by induction on $M \rhd_r N$ using lemma 8, for case $\rhd_{\beta\eta}$ use the results for $\rhd_\beta$ (lemma 8) and $\rhd_\eta$, case $\rhd_r^*$ is by induction on the length of $M \rhd_r^* N$ using the result for case $\rhd_r$.
2. Similar to 1.
3. By lemma 56.1, lemma 8 and 2 above, $M^- \succ N$.

4. Similar to 3.

5. Note that, by lemma 56.1, $FV(M) = FV(N)$ and $FV(P) = FV(Q)$.

6. Case $r \in \{\eta\}$ and $\succ = \rhd$ is by induction on M using lemmas 3.6b and 3.7. For case $\rhd_{\beta\eta}$ use the results for $\rhd_\beta$ (lemma 8) and $\rhd_\eta$. Case $\rhd_r^*$ is by induction on the length of $M \rhd_r^* N$ using the result for case $\rhd_r$.

7. Case $r \in \{\eta\}$ and $\succ = \rhd$ is by induction on M. For case $\rhd_{\beta\eta}$ use the results for $\rhd_\beta$ (lemma 8) and $\rhd_\eta$. Case $\rhd_r^*$ is by induction on the length of $M \rhd_r^* N$ using the result for case $\rhd_r$.

8. Use 6 and 7.

□

The next lemma says that there are no blocked η-redexes in a typable term.

Lemma 58 (No η-redexes are blocked in typable terms) *Let $i \in \{1, 2\}$ and $M : \langle \Gamma \vdash_i U \rangle$. If $\lambda x^n.Nx^n$ is a subterm of M, then $d(N) \le n$ and hence if $x^n \notin FV(N)$ then $\lambda x^n.Nx^n \rhd_\eta N$.*

Proof Since Nx^n is a subterm of M, by lemma 22.5, Nx^n is typable. By induction on the typing of Nx^n. We consider only the rule $\to_e$:

- Case $\vdash_1$. Let $\dfrac{N : \langle \Delta \vdash_1 T_1 \to T_2 \rangle \quad x^n : \langle (x^n : T_1) \vdash_1 T_1 \rangle}{Nx^n : \langle \Delta, (x^n : T_1) \vdash_1 T_2 \rangle}$ (by lemma 25.1).
  By lemma 23 and lemma 12.1a, $\mathrm{d}(N) = \mathrm{d}(T_1 \to T_2) = \mathrm{d}(T_2) \le \mathrm{d}(T_1) = n$. Hence $\lambda x^n.Nx^n \rhd_\eta N$.

- Case $\vdash_2$. $\dfrac{N : \langle \Gamma_1 \vdash_2 V \to T \rangle \quad x^n : \langle (x^n : W) \vdash_2 V \rangle}{Nx^n : \langle \Gamma_1, (x^n : W) \vdash_2 T \rangle}$.
  By lemma 26.1, $W \sqsubseteq V$ and, by lemma 23, $\mathrm{d}(N) = \mathrm{d}(V \to T) = 0 \le \mathrm{d}(V) = n$. Hence, if $x^n \notin FV(N)$ then $\lambda x^n.Nx^n \rhd_\eta N$.

□

The next lemma shows that the substitution lemma for $\vdash_1$, and subject reduction for η using $\vdash_1$ fail.

Lemma 59 (Subject η-reduction fails for $\vdash_1$) *Let a, b, c be different elements of $\mathcal{A}$. We have:*

1. $\lambda x^0.y^0x^0 \rhd_\eta y^0$.

2. $\lambda x^0.y^0x^0 : \langle y^0 : (a \to b) \sqcap (a \to c) \vdash_1 a \to (b \sqcap c) \rangle$.

3. *It is not possible that*
   $y^0 : \langle y^0 : (a \to b) \sqcap (a \to c) \vdash_1 a \to (b \sqcap c) \rangle$.
   Hence, subject reduction for η using $\vdash_1$ fails.

Proof 1 and 2 are easy. For 3, assume $y^0 : \langle y^0 : (a \to b) \sqcap (a \to c) \vdash_1 a \to (b \sqcap c) \rangle$. By lemma 25.1, $y^0 : (a \to b) \sqcap (a \to c) = y^0 : a \to (b \sqcap c)$ and hence, $(a \to b) \sqcap (a \to c) = a \to (b \sqcap c)$. Absurd. □

Lemma 60 (Extra Generation for $\vdash_2$)

1. *If $Mx^n : \langle \Gamma, x^n : U \vdash_2 V\rangle$, $d(V) = 0$ and $x^n \notin FV(M)$, then $V = \sqcap_{i=1}^k T_i$ where $k \geq 1$ and $\forall 1 \leq i \leq k$, $M : \langle \Gamma \vdash_2 U \to T_i\rangle$.*

2. *If $\lambda x^n.Mx^n : \langle \Gamma \vdash_2 U\rangle$ and $x^n \notin FV(M)$, then $M : \langle \Gamma \vdash_2 U\rangle$.*

Proof 1. By induction on the derivation of $Mx^n : \langle \Gamma, x^n : U \vdash_2 V\rangle$. We have three cases:

- If $\dfrac{M : \langle \Gamma \vdash_2 U \to T\rangle \quad x^n : \langle x^n : V \vdash_2 U\rangle \quad \Gamma \diamond (x^n : V) \;\; V \sqsubseteq U}{Mx^n : \langle \Gamma, x^n : V \vdash_2 T\rangle}$ (using lemma 26.1 and lemma 22), then since $U \to T \sqsubseteq V \to T$, we have $M : \langle \Gamma \vdash_2 V \to T\rangle$.

- If $\dfrac{Mx^n : \langle \Gamma, x^n : U \vdash_2 U_1\rangle \;\; Mx^n : \langle \Gamma, x^n : U \vdash_2 U_2\rangle}{Mx^n : \langle \Gamma, x^n : U \vdash_2 U_1 \sqcap U_2\rangle}$, by lemma 23 $U_1 \sqcap U_2$ is good and, by lemma 12.1b, $\mathrm{d}(U_1) = \mathrm{d}(U_2) = 0$. By IH, $U_1 = \sqcap_{i=1}^k T_i$, $U_2 = \sqcap_{i=k+1}^{k+l} T_i$, where $k, l \geq 1$ and $\forall\ 1 \leq i \leq k + l$, $M : \langle \Gamma \vdash_2 U \to T_i\rangle$.

- If $\dfrac{Mx^n : \langle \Gamma, x^n : U \vdash_2 V\rangle \quad \langle \Gamma, x^n : U \vdash_2 V\rangle \sqsubseteq \langle \Gamma', x^n : U' \vdash_2 V'\rangle}{Mx^n : \langle \Gamma', x^n : U' \vdash_2 V'\rangle}$,
by lemma 21, $\Gamma' \sqsubseteq \Gamma$, $U' \sqsubseteq U$ and $V \sqsubseteq V'$. By lemma 21, $\mathrm{d}(V) = \mathrm{d}(V') = 0$. By IH, $V = \sqcap_{i=1}^k T_i$ where $k \geq 1$ and $\forall 1 \leq i \leq k$, $M : \langle \Gamma \vdash_2 U \to T_i\rangle$. By lemma 21.4 (since V is good by lemma 23), $V' = \sqcap_{i=1}^p T'_i$ where $1 \leq p \leq k$ and $\forall 1 \leq i \leq p$, $T_i \sqsubseteq T'_i$. Since for any $1 \leq i \leq p$, $\langle \Gamma \vdash_2 U \to T_i\rangle \sqsubseteq \langle \Gamma' \vdash_2 U' \to T'_i\rangle$, then $\forall 1 \leq i \leq p$, $M : \langle \Gamma' \vdash_2 U' \to T'_i\rangle$.

2. By lemma 23, $m = \mathrm{d}(U) = \mathrm{d}(\lambda x^n.Mx^n) = \mathrm{d}(Mx^n) \leq n$. Hence $n - m \geq 0$ and $\mathrm{d}(Mx^n) = \mathrm{d}(M) = m$. By lemma 26.2, $U = \sqcap_{i=1}^k \vec{e}_{i(1:m)}(V_i \to T_i)$ where $k \geq 1$ and $\forall 1 \leq i \leq k$, $Mx^n : \langle \Gamma, x^n : \vec{e}_{i(1:m)} V_i \vdash_2 \vec{e}_{i(1:m)} T_i\rangle$.

- If $m = 0$, then $\forall 1 \leq i \leq k$, $Mx^n : \langle \Gamma, x^n : V_i \vdash_2 T_i\rangle$ and $M : \langle \Gamma \vdash_2 V_i \to T_i\rangle$ by 1. Hence, by $k - 1$ applications $\sqcap_i$, $M : \langle \Gamma \vdash_2 U\rangle$.

- If $m > 0$, then, by lemma 23 and m-applications of lemma 30.5, $\forall 1 \leq i \leq k$, $M^{-m}x^{n-m} : \langle \Gamma^{-m}, x^{n-m} : V_i \vdash_2 T_i\rangle$ and, $M^{-m} : \langle \Gamma^{-m} \vdash_2 V_i \to T_i\rangle$ by 1. Now, by m-applications of *exp*, $M : \langle \Gamma \vdash_2 \vec{e}_{i(1:m)}(V_i \to T_i)\rangle$. Finally, by k-applications of $\sqcap_i$, $M : \langle \Gamma \vdash_2 U\rangle$.

$\square$

We also show using the extra generation lemma 60 that subject reduction for η using $\vdash_2$ also holds. First, we give the basic block in the proof of subject reduction for η.

Theorem 61 *If $M : \langle \Gamma \vdash_2 U\rangle$ and $M \rhd_\eta N$, then $N : \langle \Gamma \vdash_2 U\rangle$.*

Proof By induction on the derivation of $M : \langle \Gamma \vdash_2 U\rangle$. $\to_i$ is by lemma 60.2. $\to_e$, $\sqcap_i$ and $\sqsubseteq$ are by IH. We only do the *exp* case.
Let $\dfrac{M : \langle \Gamma \vdash_2 U\rangle}{M^+ : \langle e\Gamma \vdash_2 eU\rangle}$. If $M^+ \rhd_\eta N$, then, by lemma 8.1a, $\mathrm{d}(M^+) > 0$ and, by lemma 56.1, $\mathrm{d}(N) > 0$. By lemma 8.3, $N = P^+$ and $M \rhd_\eta P$. By IH, $P : \langle \Gamma \vdash_2 U\rangle$ and, by *exp*, $N : \langle e\Gamma \vdash_2 eU\rangle$. $\square$

Corollary 62 (Subject reduction for $\beta\eta$)
If $M : \langle \Gamma \vdash_2 U\rangle$ and $M \rhd^*_{\beta\eta} N$, then $N : \langle \Gamma \vdash_2 U\rangle$.

Proof By induction on the length of the derivation of $M \rhd^*_{\beta\eta} N$ using theorems 32 and 61. $\square$

B Confluence of $\rhd^*_\beta$ and $\rhd^*_{\beta\eta}$

In this section we establish the confluence of $\rhd^*_\beta$ and $\rhd^*_{\beta\eta}$ using the standard parallel reduction method.

Definition 63 *Let $r \in \{\beta, \beta\eta\}$. We define on $\mathcal{M}$ the binary relation $\stackrel{\rho_r}{\to}$ by:*

- $M \stackrel{\rho_r}{\to} M$
- *If $M \stackrel{\rho_r}{\to} M'$ and $x^n \in FV(M)$, then $\lambda x^n.M \stackrel{\rho_\beta}{\to} \lambda x^n.M'$.*
- *If $M \stackrel{\rho_r}{\to} M'$, $N \stackrel{\rho_\beta}{\to} N'$ and $M \diamond N$ then $MN \stackrel{\rho_r}{\to} M'N'$*
- *If $M \stackrel{\rho_r}{\to} M'$, $N \stackrel{\rho_r}{\to} N'$, $M \diamond N$, $x^n \in FV(M)$ and $d(N) = n$, then $(\lambda x^n.M)N \stackrel{\rho_r}{\to} M'[x^n := N']$*
- *If $M \stackrel{\rho_{\beta\eta}}{\to} M'$, $x^n \notin FV(M)$ and $d(M) \leq n$, then $\lambda x^n.Mx^n \stackrel{\rho_{\beta\eta}}{\to} M'$*

We denote the transitive closure of $\stackrel{\rho_r}{\to}$ by $\stackrel{\rho_r}{\twoheadrightarrow}$. When $M \stackrel{\rho_r}{\to} N$ (resp. $M \stackrel{\rho_r}{\twoheadrightarrow} N$), we can also write $N \stackrel{\rho_r}{\leftarrow} M$ (resp. $N \stackrel{\rho_r}{\twoheadleftarrow} M$). If $R, R' \in \{\stackrel{\rho_r}{\to}, \stackrel{\rho_r}{\twoheadrightarrow}, \stackrel{\rho_r}{\leftarrow}, \stackrel{\rho_r}{\twoheadleftarrow}\}$, we write $M_1 R M_2 R' M_3$ instead of $M_1 R M_2$ and $M_2 R' M_3$.

Lemma 64 *Let $r \in \{\beta, \beta\eta\}$ and $M \in \mathcal{M}$.*

1. *If $M \rhd_r M'$, then $M \stackrel{\rho_r}{\to} M'$.*
2. *If $M \stackrel{\rho_r}{\to} M'$, then $M' \in \mathcal{M}$, $M \rhd^*_r M'$, $FV(M) = FV(M')$ and $d(M) = d(M')$.*
3. *If $M \stackrel{\rho_r}{\to} M'$, $N \stackrel{\rho_r}{\to} N'$ and $M \diamond N$, then $M' \diamond N'$.*

Proof 1. By induction on the derivation of $M \rhd_r M'$. 2. By induction on the derivation of $M \stackrel{\rho_r}{\to} M'$ using lemmas 6.1, 56.1, 8.11 and 57.8. 3. $M' \diamond N'$ since by 2, $FV(M) = FV(M')$ and $FV(N) = FV(N')$ and $M \diamond N$. $\square$

Lemma 65 *Let $r \in \{\beta, \beta\eta\}$, $M, N \in \mathcal{M}$, $N \stackrel{\rho_r}{\to} N'$ and $M \diamond N$. We have:*

1. *$M[x^n := N] \stackrel{\rho_r}{\to} M[x^n := N']$.*
2. *If $M \stackrel{\rho_r}{\to} M'$ and $d(N) = n$, then $M[x^n := N] \stackrel{\rho_r}{\to} M'[x^n := N']$.*

Proof 1. By induction on M using lemmas 3.2, 3.4 and 3.6a.
2. By induction on $M \stackrel{\rho_r}{\to} M'$ using 1, lemmas 3.2, 3.4, 3.6a and 64.3. We only do one interesting case where $(\lambda y^m.M_1)M_2 \stackrel{\rho_\beta}{\to} M'_1[y^m := M'_2]$, $M_1 \stackrel{\rho_\beta}{\to} M'_1$, $M_2 \stackrel{\rho_\beta}{\to} M'_2$, $M_1 \diamond M_2$, $y^m \in FV(M_1)$ and $\mathrm{d}(M_2) = m$. Then:

a. $M_1[x^n := N] \stackrel{\rho_\beta}{\to} M'_1[x^n := N']$, by IH and lemma 3.2.

b. $M_2[x^n := N] \stackrel{\rho_\beta}{\to} M'_2[x^n := N']$, by IH and lemma 3.2.

c. $M_1[x^n := N] \diamond M_2[x^n := N]$, by lemmas 3.2 and 3.4.

d. $M'_1[x^n := N'] \diamond M'_2[x^n := N']$, by a., b., c., and lemma 64.3.

e. $y^m \in FV(M_1[x^n := N])$.

f. $\mathrm{d}(M_2[x^n := N]) = m$, by lemmas 3.2.

g. $y^m \notin FV(N')$, by BC and by lemma 3.7,
$M'_1[x^n := N'][y^m := M'_2[x^n := N']] = M'_1[y^m := M'_2][x^n := N']$.

Hence, $(\lambda y^m.M_1[x^n := N])M_2[x^n := N] \stackrel{\rho_\beta}{\to} M'_1[x^n := N'][y^m := M'_2[x^n := N']]$
and so, $((\lambda y^m.M_1)M_2)[x^n := N] \stackrel{\rho_\beta}{\to} M'_1[y^m := M'_2][x^n := N']$. $\square$

Lemma 66 *Let $r \in \{\beta, \beta\eta\}$. Note that:*

1. *If $x^n \stackrel{\rho_r}{\to} N$, then $N = x^n$.*
2. *If $\lambda x^n.P \stackrel{\rho_\beta}{\to} N$, then $N = \lambda x^n.P'$ where $P \stackrel{\rho_\beta}{\to} P'$ and $x^n \in FV(P) \cap FV(P')$.*
3. *If $\lambda x^n.P \stackrel{\rho_{\beta\eta}}{\to} N$ then one of the following holds:*
   - *$N = \lambda x^n.P'$ where $P \stackrel{\rho_{\beta\eta}}{\to} P'$ and $x^n \in FV(P) \cap FV(P')$.*
   - *$P = P'x^n$ where $x^n \notin FV(P') \cup FV(N)$, $d(P') \le n$ and $P' \stackrel{\rho_{\beta\eta}}{\to} N$.*
4. *If $PQ \stackrel{\rho_r}{\to} N$, then one of the following holds:*
   - *$N = P'Q'$, $P \stackrel{\rho_r}{\to} P'$, $Q \stackrel{\rho_r}{\to} Q'$, $P \diamond Q$, and $P' \diamond Q'$.*
   - *$P = \lambda x^n.P'$, $N = P''[x^n := Q']$, $x^n \in FV(P') \cap FV(P'')$, $d(Q) = d(Q') = n$, $P' \stackrel{\rho_r}{\to} P''$, $Q \stackrel{\rho_r}{\to} Q'$, $P' \diamond Q$ and $P'' \diamond Q'$.*

Proof 1. By induction on the derivation of $x^n \stackrel{\rho_r}{\to} N$.
2. By induction on the derivation of $\lambda x^n.P \stackrel{\rho_\beta}{\to} N$ using lemma 64.2.
3. By induction on the derivation of $\lambda x^n.P \stackrel{\rho_{\beta\eta}}{\to} N$ using lemma 64.2.
4. By induction on the derivation of $PQ \stackrel{\rho_r}{\to} N$ using lemma 64.2 and 64.3. $\square$

Lemma 67 *Let $r \in \{\beta, \beta\eta\}$ and $M, M_1, M_2 \in \mathcal{M}$.*

1. *If $M_2 \stackrel{\rho_r}{\leftarrow} M \stackrel{\rho_r}{\to} M_1$, then there is $M' \in \mathcal{M}$ such that $M_2 \stackrel{\rho_r}{\to} M' \stackrel{\rho_r}{\leftarrow} M_1$.*
2. *If $M_2 \stackrel{\rho_r}{\twoheadleftarrow} M \stackrel{\rho_r}{\twoheadrightarrow} M_1$, then there is $M' \in \mathcal{M}$ such that $M_2 \stackrel{\rho_r}{\twoheadrightarrow} M' \stackrel{\rho_r}{\twoheadleftarrow} M_1$.*

Proof 1. Both cases by induction on M. We only do the $\beta\eta$ case making discriminate use of lemma 66.

- If $M = x^n$, by lemma 66, $M_1 = M_2 = x^n$. Take $M' = x^n$.
- If $N_2P_2 \stackrel{\rho_{\beta\eta}}{\leftarrow} NP \stackrel{\rho_{\beta\eta}}{\to} N_1P_1$ where $N_2 \stackrel{\rho_{\beta\eta}}{\leftarrow} N \stackrel{\rho_{\beta\eta}}{\to} N_1$, $P_2 \stackrel{\rho_{\beta\eta}}{\leftarrow} P \stackrel{\rho_{\beta\eta}}{\to} P_1$, $N \diamond P$, $N_1 \diamond P_1$ and $N_2 \diamond P_2$, then, by IH, $\exists N', P'$ such that $N_2 \stackrel{\rho_{\beta\eta}}{\to} N' \stackrel{\rho_{\beta\eta}}{\leftarrow} N_1$ and $P_2 \stackrel{\rho_{\beta\eta}}{\to} P' \stackrel{\rho_{\beta\eta}}{\leftarrow} P_1$. Hence, $N_2P_2 \stackrel{\rho_{\beta\eta}}{\to} N'P' \stackrel{\rho_{\beta\eta}}{\leftarrow} N_1P_1$.
- If $(\lambda x^n.P_1)Q_1 \stackrel{\rho_{\beta\eta}}{\leftarrow} (\lambda x^n.P)Q \stackrel{\rho_{\beta\eta}}{\to} P_2[x^n := Q_2]$ where $\lambda x^n.P \stackrel{\rho_{\beta\eta}}{\to} \lambda x^n.P_1$, $P \stackrel{\rho_{\beta\eta}}{\to} P_2$, $Q_1 \stackrel{\rho_{\beta\eta}}{\leftarrow} Q \stackrel{\rho_{\beta\eta}}{\to} Q_2$, $\lambda x^n.P_1 \diamond Q_1$, $\lambda x^n.P \diamond Q$, $P \diamond Q$, $P_2 \diamond Q_2$, $x^n \in FV(P) \cap FV(P_2)$, $\mathrm{d}(Q) = \mathrm{d}(Q_2) = n$, then, by lemma 66, $x^n \in FV(P) \cap FV(P_1)$ and $P \stackrel{\rho_{\beta\eta}}{\to} P_1$. By IH, $\exists P', Q'$ such that $P_1 \stackrel{\rho_{\beta\eta}}{\to} P' \stackrel{\rho_{\beta\eta}}{\leftarrow} P_2$ and $Q_1 \stackrel{\rho_{\beta\eta}}{\to} Q' \stackrel{\rho_{\beta\eta}}{\leftarrow} Q_2$. By lemma 64.2, $\mathrm{d}(Q_1) = \mathrm{d}(Q) = n$. By lemma 3.2, $P_1 \diamond Q_1$. Hence, $(\lambda x^n.P_1)Q_1 \stackrel{\rho_{\beta\eta}}{\to} P'[x^n := Q']$.

  Moreover, since $P_2 \stackrel{\rho_{\beta\eta}}{\to} P'$, $Q_2 \stackrel{\rho_{\beta\eta}}{\to} Q'$, $P_2 \diamond Q_2$, and $\mathrm{d}(Q_2) = n$, then, by lemma 65.2, $P_2[x^n := Q_2] \stackrel{\rho_{\beta\eta}}{\to} P'[x^n := Q']$.
- If $P_1[x^n := Q_1] \stackrel{\rho_{\beta\eta}}{\leftarrow} (\lambda x^n.P)Q \stackrel{\rho_{\beta\eta}}{\to} P_2[x^n := Q_2]$ where $P_1 \stackrel{\rho_{\beta\eta}}{\leftarrow} P \stackrel{\rho_{\beta\eta}}{\to} P_2$, $P \diamond Q$, $P_1 \diamond Q_1$, $P_2 \diamond Q_2$, $Q_1 \stackrel{\rho_{\beta\eta}}{\leftarrow} Q \stackrel{\rho_{\beta\eta}}{\to} Q_2$, $x^n \in FV(P) \cap FV(P_1) \cap FV(P_2)$ and $\mathrm{d}(Q) = \mathrm{d}(Q_1) = \mathrm{d}(Q_2) = n$, then, by IH, $\exists P', Q'$ where $P_1 \stackrel{\rho_{\beta\eta}}{\to} P' \stackrel{\rho_{\beta\eta}}{\leftarrow} P_2$ and $Q_1 \stackrel{\rho_{\beta\eta}}{\to} Q' \stackrel{\rho_{\beta\eta}}{\leftarrow} Q_2$. Hence, by lemma 65.2,
  $P_1[x^n := Q_1] \stackrel{\rho_{\beta\eta}}{\to} P'[x^n := Q'] \stackrel{\rho_{\beta\eta}}{\leftarrow} P_2[x^n := Q_2]$.
- If $\lambda x^n.N_2 \stackrel{\rho_{\beta\eta}}{\leftarrow} \lambda x^n.N \stackrel{\rho_{\beta\eta}}{\to} \lambda x^n.N_1$ where $N_2 \stackrel{\rho_{\beta\eta}}{\leftarrow} N \stackrel{\rho_{\beta\eta}}{\to} N_1$ and $x^n \in FV(N_1) \cap FV(N_2) \cap FV(N)$, by IH, there is N' such that $N_2 \stackrel{\rho_{\beta\eta}}{\to} N' \stackrel{\rho_{\beta\eta}}{\leftarrow} N_1$. Hence, $\lambda x^n.N_2 \stackrel{\rho_{\beta\eta}}{\to} \lambda x^n.N' \stackrel{\rho_{\beta\eta}}{\leftarrow} \lambda x^n.N_1$.

- If $M_1 \stackrel{\rho_{\beta\eta}}{\leftarrow} \lambda x^n.Px^n \stackrel{\rho_{\beta\eta}}{\to} M_2$, $x^n \notin FV(P) \cup FV(M_1) \cup FV(M_2)$ and $\mathrm{d}(P) \leq n$, then, by IH, there is M' such that $M_2 \stackrel{\rho_{\beta\eta}}{\to} M' \stackrel{\rho_{\beta\eta}}{\leftarrow} M_1$.

- If $M_1 \stackrel{\rho_{\beta\eta}}{\leftarrow} \lambda x^n.Px^n \stackrel{\rho_{\beta\eta}}{\to} \lambda x^n.P'$, where $P \stackrel{\rho_{\beta\eta}}{\to} M_1$, $Px^n \stackrel{\rho_{\beta\eta}}{\to} P'$, $\mathrm{d}(P) \leq n$ and $x^n \notin FV(P) \cup FV(M_1)$. By lemma 66, $P' = P''x^n$ and $P \stackrel{\rho_{\beta\eta}}{\to} P''$. By IH, there is M' such that $P'' \stackrel{\rho_{\beta\eta}}{\to} M' \stackrel{\rho_{\beta\eta}}{\leftarrow} M_1$. By lemma 64.2, $x^n \notin FV(P'')$ and $\mathrm{d}(P'') \leq n$. Hence, $M_2 = \lambda x^n.P''x^n \stackrel{\rho_{\beta\eta}}{\to} M' \stackrel{\rho_{\beta\eta}}{\leftarrow} M_1$.

2. First show by induction on $M \stackrel{\rho_r}{\twoheadrightarrow} M_1$ (and using 1) that if $M_2 \stackrel{\rho_r}{\leftarrow} M \stackrel{\rho_r}{\twoheadrightarrow} M_1$, then there is M' such that $M_2 \stackrel{\rho_r}{\twoheadrightarrow} M' \stackrel{\rho_r}{\leftarrow} M_1$. Then use this to show 2 by induction on $M \stackrel{\rho_r}{\twoheadrightarrow} M_2$. □

Theorem 68 (Confluence of $\rhd^*_\beta/\rhd^*_{\beta\eta}$) *Let $r \in \{\beta, \beta\eta\}$, $M, M_1, M_2 \in \mathcal{M}$.*

1. *If $M \rhd^*_r M_1$ and $M \rhd^*_r M_2$, then there is M' such that $M_1 \rhd^*_r M'$ and $M_2 \rhd^*_r M'$.*

2. *$M_1 \simeq_\beta M_2$ iff there is a term M such that $M_1 \rhd^*_\beta M$ and $M_2 \rhd^*_\beta M$.*

Proof

1. By lemma 67.2, $\stackrel{\rho_r}{\twoheadrightarrow}$ is confluent. By lemma 64.1 and 64.2, $M \stackrel{\rho_r}{\twoheadrightarrow} N$ iff $M \rhd^*_r N$. Then $\rhd^*_r$ is confluent.

2. If) is by definition of $\simeq_\beta$. Only if) is by induction on $M_1 \simeq_\beta M_2$ using 1.

□

C Strong normalization of the type systems $\vdash_1$ and $\vdash_2$

To show the strong normalisation of our two type systems, we will use the well-known reducibility method. First, we define the sets of strongly normalising terms for each degree n.

Definition 69 1. *We say that $M \in \mathcal{M}$ is strongly normalising if there are no infinite derivations $M \rhd_\beta M_1 \rhd_\beta \ldots$.*

2. *For every $n \in \mathbb{N}$, we define $\mathbb{SN}^n = \{M \in \mathcal{M}^n \; / \; M$ is strongly normalizing$\}$ and $\mathbb{SN} = \bigcup_{i \in \mathbb{N}} \mathbb{SN}^i$. Note that $\mathbb{SN} = \{M \in \mathcal{M} \; / \; M$ is strongly normalizing$\}$ and $\mathbb{SN}^n = \mathbb{SN} \cap \mathcal{M}^n$.*

Remark 70 *We can show that since we work with the $\lambda I^{\mathbb{N}}$-calculus, strong normalisation is equivalent to weak normalisation. However, since this result is not needed for this paper, we do not discuss it further.*

The next lemma shows that some terms are strongly normalising if they are parts or β-expansions of strongly normalising terms.

Lemma 71 1. *If $Mx^n \in \mathbb{SN}$, then $M \in \mathbb{SN}$.*

2. *Let $d(N) = n$. If $M[x^n := N] \in \mathbb{SN}$, then $M \in \mathbb{SN}$.*

3. *Let $k \geq 0$ and $d(N) = n$. If $M, N, N_1, ..., N_k, M[x^n := N]\ N_1...N_k \in \mathbb{SN}$, then $(\lambda x^n.M)NN_1...N_k \in \mathbb{SN}$.*

Proof 1. If $M \rhd_\beta M_1 \rhd_\beta \dots$ is an infinite derivation, then $Mx^n \rhd_\beta M_1 x^n \rhd_\beta \dots$ is an infinite derivation. Absurd.
2. Since $M[x^n := N] \in \mathcal{M}$, then $M \diamond N$ by BC and lemma 3.2. If $M \rhd_\beta M_1 \rhd_\beta \dots$ is an infinite derivation, then, by lemma 8.8, $M_i \diamond N$ for every $i \geq 1$. By lemma 8.9, $M[x^n := N] \rhd_\beta M_1[x^n := N] \rhd_\beta \dots$ is an infinite derivation. Absurd.
3. As $M[x^n := N] \in \mathcal{M}$, then $M \diamond N$ by BC and lemma 3.2. Since $M, N, N_1, \dots, N_k \in \mathbb{SN}$, any infinite $\rhd_\beta$-derivation starting at $(\lambda x^n.M)NN_1 \dots N_k$ has the form: $(\lambda x^n.M)N \rhd^*_\beta M'[x^n := N']N'_1 \dots N'_k \rhd_\beta \dots$ where $M \rhd^*_\beta M'$, $N \rhd^*_\beta N'$, $\forall\, 1 \leq i \leq k$, $N_i \rhd^*_\beta N'_i$ and the infinite derivation continues from $M'[x^n := N']N'_1 \dots N'_k$. By lemma 8.11, $M[x^n := N] \rhd^*_\beta M'[x^n := N']$.
Hence $M[x^n := N]N_1 \dots N_k \rhd^*_\beta M'[x^n := N']N'_1 \dots N'_k$ and there is an infinite derivation starting at $M[x^n := N]N_1 \dots N_k \in \mathbb{SN}$. Absurd. □

Strong normalisation is closed under the lifting and decreasing of the degree of a term.

Lemma 72 *1. $M \in \mathbb{SN}$ iff $M^+ \in \mathbb{SN}$.*

2. If $d(M) > 0$, $M \in \mathbb{SN}$ iff $M^- \in \mathbb{SN}$.

3. For every $n \in \mathbb{N}$, $(\mathbb{SN}^n)^+ = \mathbb{SN}^{n+1}$ and $(\mathbb{SN}^{n+1})^- = \mathbb{SN}^n$.

Proof 1. Only if): Let $M \in \mathbb{SN}$. If $M^+ \notin \mathbb{SN}$, take an infinite derivation $M^+ \rhd_\beta N_1 \rhd_\beta N_2 \rhd_\beta \dots$. By lemma 8.1a and lemma 6.1, $\mathrm{d}(M^+) > 0$ and $\forall i \geq 1$, $\mathrm{d}(N_i) > 0$. By lemmas 8.3 and 8.1b, $\forall i \geq 1$, $\exists M_i = N_i^-$ such that $N_i = M_i^+$ and $M \rhd_\beta M_1 \rhd_\beta M_2 \rhd_\beta \dots$ is an infinite derivation. Absurd.
If): Let $M^+ \in \mathbb{SN}$. If $M \notin \mathbb{SN}$, take an infinite derivation $M \rhd_\beta M_1 \rhd_\beta M_2 \rhd_\beta \dots$. By lemma 8.4, $M^+ \rhd_\beta M_1^+ \rhd_\beta M_2^+ \rhd_\beta \dots$ is an infinite derivation. Absurd.
2. By lemma 8.3, $M = (M^-)^+$. By 1. $M^- \in \mathbb{SN}$ iff $M = (M^-)^+ \in \mathbb{SN}$.
3. Use 1. and 2. □

Now we define SN-saturated sets and establish some of their properties.

Definition 73 *We say that $\mathcal{X} \subseteq \mathcal{M}$ is SN-saturated iff whenever $M, N, N_1, ..., N_k \in \mathbb{SN}$, $x^n \in FV(M)$, $d(N) = n$ and $M[x^n := N]N_1...N_k \in \mathcal{X}$, then $(\lambda x^n.M)NN_1...N_k \in \mathcal{X}$.*

Lemma 74 *1. For every $n \in \mathbb{N}$, the set $\mathbb{SN}^n$ is SN-saturated.*

2. If $\mathcal{X}, \mathcal{Y}$ are SN-saturated sets, then $\mathcal{X} \cap \mathcal{Y}$ is SN-saturated.

3. If $\mathcal{X}$ is SN-saturated, then $\mathcal{X}^+$ is SN-saturated.

4. If $\mathcal{Y}$ is SN-saturated, then, for every set $\mathcal{X} \subseteq \mathbb{SN}$, $\mathcal{X} \rightsquigarrow \mathcal{Y}$ is SN-saturated.

Proof

1. Let $M, N, N_1, ..., N_k \in \mathbb{SN}$ such that $\mathrm{d}(N) = m$, $x^m \in FV(M)$ and $M[x^m := N]N_1...N_k \in \mathbb{SN}^n$. Since $(\lambda x^m.M)NN_1...N_k \rhd_\beta M[x^m := N]N_1...N_k$ and $\mathrm{d}(M[x^m := N]N_1...N_k) = n$, then, by lemma 6.1, $\mathrm{d}((\lambda x^m.M)NN_1...N_k) = n$. Since $M, N, N_1, ..., N_k, M[x^m := N]N_1...N_k \in \mathbb{SN}$, then, by lemma 71, $(\lambda x^m.M)NN_1...N_k \in \mathbb{SN}$.

2. Easy.

3. If $M, N, N_1, ..., N_k \in \mathbb{SN}$, $x^n \in FV(M)$, $\mathrm{d}(N) = n$ and $M[x^n := N]\ N_1...N_k \in \mathcal{X}^+$, then, by lemma 8.1(c)iii, $(M[x^n := N])^- N_1^- ... N_k^- \in \mathcal{X}$. Since $M \diamond N$ (by lemma 3), $n > 0$ and for each $P \in \{M, N, N_1, \dots, N_k\}$, $\mathrm{d}(P) > 0$, then, by lemma 8.3 $(M[x^n := N])^- = M^-[x^{n-1} := N^-]$ and, by lemma 72.2, $M^-, N^-, N_1^-, ..., N_k^- \in \mathbb{SN}$. Moreover, $x^{n-1} \in FV(M^-)$ and by lemma 8.3 $\mathrm{d}(N^-) = n - 1$. Hence since $\mathcal{X}$ is SN-saturated, $(\lambda x^{n-1}.M^-)N^- N_1^- ... N_k^- \in \mathcal{X}$ and, by lemma 8.3, $(\lambda x^n.M)NN_1...N_k \in \mathcal{X}^+$.

4. If $M, N, N_1, ..., N_k \in \mathbb{SN}$, $x^n \in FV(M)$, $\mathrm{d}(N) = n$, $M[x^n := N]\ N_1...N_k \in \mathcal{X} \rightsquigarrow \mathcal{Y}$ and $P \in \mathcal{X} \subseteq \mathbb{SN}$ such that $(\lambda x^n.M)NN_1...N_k \diamond P$, then, by lemma 8.8 $M[x^n := N]N_1...N_k \diamond P$ and hence $M[x^n := N]\ N_1...N_kP \in \mathcal{Y}$. Since $\mathcal{Y}$ is SN-saturated, $(\lambda x^n.M)NN_1...N_kP \in \mathcal{Y}$.
Thus $(\lambda x^n.M\ N)\ N_1...N_k \in \mathcal{X} \rightsquigarrow \mathcal{Y}$.

□

Now we define the so-called SN-interpretation of types.

Definition 75 *1. Let $x \in \mathcal{V}$ and $n \in \mathbb{N}$. We define*
$\mathbb{SN}^n_x = \{x^n\ N_1...N_k \in \mathcal{M}\ /\ k \geq 0$ *and* $\forall 1 \leq i \leq k,\ N_i \in \mathbb{SN}^{m_i}$ *and* $m_i \geq n\}$.

2. An SN-interpretation $\mathcal{I} : \mathcal{A} \mapsto \mathcal{P}(\mathcal{M}^0)$ is a function such that for all $a \in \mathcal{A}$:
• $\mathcal{I}(a)$ is SN-saturated • $\forall x \in \mathcal{V}$, $\mathbb{SN}^0_x \cap \mathbb{M} \subseteq \mathcal{I}(a) \subseteq \mathbb{SN}^0 \cap \mathbb{M}$

3. We extend an SN-interpretation $\mathcal{I}$ to $\mathcal{T}$ (hence also to $\mathbb{U}$) as follows:
• $\mathcal{I}(eU) = \mathcal{I}(U)^+$ • $\mathcal{I}(U_1 \sqcap U_2) = \mathcal{I}(U_1) \cap \mathcal{I}(U_2)$ • $\mathcal{I}(U \to T) = \mathcal{I}(U) \rightsquigarrow \mathcal{I}(T)$.

Lemma 76 *1. For every $x \in \mathcal{V}$, $\mathbb{SN}^n_x \subseteq \mathbb{SN}^n$.*

2. On $\mathcal{T}$ (and hence on $\mathbb{U}$), we have:
If U is a type and $\mathcal{I}$ is an SN-interpretation then:

(a) $\mathcal{I}(U)$ is SN-saturated.

(b) Let $x \in \mathcal{V}$. If U is good and $d(U) = n$, then $x^n \in \mathbb{SN}^n_x \cap \mathbb{M} \subseteq \mathcal{I}(U) \subseteq \mathbb{SN}^n \cap \mathbb{M}$.

Proof

1. For every $x \in \mathcal{V}$ and for every $M \in \mathbb{SN}^n_x$, $\mathrm{d}(M) = n$ and $M \in \mathbb{SN}$, then $M \in \mathbb{SN}^n$.

2a. By induction on U type using lemma 74.

2b. Obviously, $x^n \in \mathbb{SN}^n_x \cap \mathbb{M}$. We show $\mathbb{SN}^n_x \cap \mathbb{M} \subseteq \mathcal{I}(U) \subseteq \mathbb{SN}^n \cap \mathbb{M}$ by induction on U good.
Case $U = a$: by definition. Case $U = U_1 \sqcap U_2$ (resp. $U = eV$): use IH since, by lemma 12, U_1, U_2 are good and $\mathrm{d}(U_1) = \mathrm{d}(U_2)$ (resp. V is good, $\mathrm{d}(U) = \mathrm{d}(V) + 1$, $(\mathbb{SN}^n_x)^+ = \mathbb{SN}^{n+1}_x$ and, by lemma 72.3 $(\mathbb{SN}^n)^+ = \mathbb{SN}^{n+1}$).
Case $U = V \to T$: by lemma 12, V, T are good and $m = \mathrm{d}(V) \geq \mathrm{d}(T) = \mathrm{d}(U) = n$.

   - Let $k \geq 0$ and $x^n N_1...N_k \in \mathbb{SN}^n_x \cap \mathbb{M}$ where $\forall 1 \leq i \leq k$, $N_i \in \mathbb{SN}^{m_i}$ and $m_i \geq n$. Take $N \in \mathcal{I}(V)$ such that $(x^n N_1...N_k) \diamond N$ (hence $x^n N_1...N_k N \in \mathcal{M}$). By IH, $\mathcal{I}(V) \subseteq \mathbb{SN}^m \cap \mathbb{M}$ and $\mathrm{d}(N) = m \geq n$. Again, by IH, $x^n N_1...N_k N \in \mathbb{SN}^n_x \cap \mathbb{M} \subseteq \mathcal{I}(T)$. Thus $x^n N_1...N_k \in \mathcal{I}(V \to T)$.
   - Let $M \in \mathcal{I}(V \to T)$ and $x \in \mathcal{V}$ such that $\forall p \in \mathbb{N}$, $x^p \notin FV(M)$. Hence, $M \diamond x^m$. Since $x^m \in \mathbb{SN}^m_x \cap \mathbb{M}$, by IH, $x^m \in \mathcal{I}(V)$. Then $Mx^m \in \mathcal{I}(T) \subseteq \mathbb{SN}^n \cap \mathbb{M}$ by IH, and $\mathrm{d}(Mx^m) = \min(\mathrm{d}(M), m) = n$. Since $Mx^m \in \mathbb{M}$, by lemma 2, M is good and $\mathrm{d}(M) \leq m$. Thus $\mathrm{d}(M) = n$ and, by lemma 71.1, $M \in \mathbb{SN}^n \cap \mathbb{M}$.

□

Lemma 77 *If $\mathcal{I}$ be an SN-interpretation and $U \sqsubseteq V$, then $\mathcal{I}(U) \subseteq \mathcal{I}(V)$.*

Proof By induction on the derivation of $U \sqsubseteq V$. □

Lemma 78 *Let $i \in \{1,2\}$ and $M : \langle (x_i^{n_i} : U_i)_n \vdash_i U \rangle$, $\mathcal{I}$ be an SN-interpretation and $\forall 1 \leq i \leq n$, $N_i \in \mathcal{I}(U_i)$. If $M[(x_i^{n_i} := N_i)_n] \in \mathcal{M}$, then $M[(x_i^{n_i} := N_i)_n] \in \mathcal{I}(U)$.*

Proof By induction on the derivation of $M : \langle (x_i^{n_i} : U_i)_n \vdash_i U \rangle$. The proof is similar to that of lemma 38. We use lemma 77 for the typing rule $\sqsubseteq$. We only look at the typing rule $\to_i$.

Let $\dfrac{M : \langle (x_i^{n_i} : U_i)_n, x^m : V \vdash_i T \rangle}{\lambda x^m.M : \langle (x_i^{n_i} : U_i)_n \vdash_i V \to T \rangle}$ where $(\lambda x^m.M)[(x_i^{n_i} := N_i)_n] \in \mathcal{M}$ and $\forall\, 1 \leq i \leq n$, $N_i \in \mathcal{I}(U_i)$. Let $N \in \mathcal{I}(V)$ where $(\lambda x^m.M)[(x_i^{n_i} := N_i)_n] \diamond N$.
Since $(\lambda x^m.M[(x_i^{n_i} := N_i)_n]) \diamond N$, by lemma 3, $M[(x_i^{n_i} := N_i)_n] \diamond N$ and $M[(x_i^{n_i} := N_i)_n][x^m := N] = M[(x_i^{n_i} := N_i)_n, x^m := N] \in \mathcal{M}$. By IH, $M[(x_i^{n_i} := N_i)_n, x^m := N] \in \mathcal{I}(T)$. By lemma 12, V, T are good and $\mathrm{d}(V) = m$. Hence, by lemma 76, $\mathrm{d}(N) = m$ and $(\lambda x^m.M[(x_1^{n_1} := N_1)_n])N \rhd_\beta M[(x_i^{n_i} := N_i)_n, x^m := N] \in \mathcal{I}(T)$. By lemma 76, $\mathcal{I}(T), \mathcal{I}(V) \subseteq \mathbb{SN}$ and so N, $M[(x_i^{n_i} := N_i)_n, x^m := N] \in \mathbb{SN}$. Hence $M[(x_i^{n_i} := N_i)_n] \in \mathbb{SN}$ by lemma 71, and since $\mathcal{I}(T)$ is SN-saturated (lemma 76), we get by definition 73 that $(\lambda x^m.M[(x_1^{n_1} := N_1)_n])N \in \mathcal{I}(T)$. Hence, $\lambda x^m.M[(x_i^{n_i} := N_i)_n] \in \mathcal{I}(V) \rightsquigarrow \mathcal{I}(T)$. □

Theorem 79 *Let $i \in \{1,2\}$. If $M : \langle (x_i^{n_i} : U_i)_n \vdash_i U \rangle$, then M is strongly normalizing.*

Proof Let $\mathrm{d}(U) = k$ and take, by lemmas 74.1 and 76.1, the SN-interpretation $\mathcal{I}$ defined by: for all $a \in \mathcal{A}$, $\mathcal{I}(a) = \mathbb{SN}^0$. By lemma 23, U is good and $\forall 1 \leq i \leq n$, U_i is good and $\mathrm{d}(U_i) = n_i$. By lemma 76, $x_i^{n_i} \in \mathbb{SN}_{x_i}^{n_i} \cap \mathbb{M} \subseteq \mathcal{I}(U_i)$ $\forall 1 \leq i \leq n$ and $\mathcal{I}(U) \subseteq \mathbb{SN}^k$. Hence, by lemma 78, $M = M[(x_i^{n_i} := x_i^{n_i})_n] \in \mathcal{I}(U) \subseteq \mathbb{SN}^k$ and M is strongly normalizing. □

D Removing indices from $\vdash_1$

We assume familiarity with the λI-calculus (the $\lambda I^{\mathbb{N}}$-calculus without indices). We use the same syntax of types for the λI-calculus and we define λI-environments to be exactly those of the $\lambda I^{\mathbb{N}}$-calculus but where all upper indices disappear from the variables. We use the same meta-variables in the λI- and $\lambda I^{\mathbb{N}}$-calculi. If Γ_1 and Γ_2 are two λI-environments, then we define $\Gamma_1 \sqcap \Gamma_2$ as usual. Moreover, if $\Gamma = (x_i : T_i)_n$ is a λI-environment, then we define $e\Gamma = (x_i : eT_i)_n$.

Definition 80 *1. We define the very good types on $\mathcal{T}$ by:*

- *If $a \in \mathcal{A}$, then a is very good.*
- *If U, T are very good and $d(U) = d(T)$, then $U \to T$ and $U \sqcap T$ are very good.*
- *If U is very good and $e \in \mathcal{E}$, then eU is very good.*

Note that if U is very good then U is very good.

2. We define $\overset{\circ}{\vdash}$ to be the typing relation on the $\lambda I^{\mathbb{N}}$-calculus given by all the rules of $\vdash_1$ except for ax which is replaced by:

$$\frac{T \textit{ very good} \qquad d(T) = n}{x^n : \langle (x^n : T) \vdash_1 T \rangle} \; ax^\circ$$

Definition 81 *1. Let $r \in \{s, v\}$. We define the typing system $\vdash_r$ for the λI-calculus, based on the rules $\{ax_r, \to_{ir}, \to_{er}, \sqcap_{ir}, exp_r\}$ given as follows:*

$$\frac{}{x : \langle (x : T) \vdash_s T \rangle} \; ax_s \qquad\qquad \frac{T \textit{ very good}}{x : \langle (x : T) \vdash_v T \rangle} \; ax_v$$

$$\frac{M : \langle \Gamma, (x : T_1) \vdash_r T_2 \rangle}{\lambda x.M : \langle \Gamma \vdash_r T_1 \to T_2 \rangle} \quad \to_{i_r}$$

$$\frac{M_1 : \langle \Gamma_1 \vdash_r T_1 \to T_2 \rangle \quad M_2 : \langle \Gamma_2 \vdash_r T_1 \rangle}{M_1\ M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_r T_2 \rangle} \quad \to_{e_r}$$

$$\frac{M : \langle \Gamma_1 \vdash_s T_1 \rangle \quad M : \langle \Gamma_2 \vdash_s T_2 \rangle}{M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_s T_1 \sqcap T_2 \rangle} \quad \sqcap_{is}$$

$$\frac{M : \langle \Gamma_1 \vdash_v T_1 \rangle \quad M : \langle \Gamma_2 \vdash_v T_2 \rangle \quad d(T_1) = d(T_2)}{M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_v T_1 \sqcap T_2 \rangle} \quad \sqcap_{iv}$$

$$\frac{M : \langle \Gamma \vdash_r T \rangle}{M : \langle e\Gamma \vdash_r eT \rangle} \quad exp_r$$

2. *We associate to each $\lambda I^{\mathbb{N}}$-term M a λI-term $\overline{M}$ by induction as follows:*
$\overline{x^n} = x$ $\qquad \overline{M_1\ M_2} = \overline{M_1}\ \overline{M_2}$ $\qquad \overline{\lambda x^n.M} = \lambda x.\overline{M}$

3. *If $\Gamma = (x_i^{n_i} : T_i)_k$, then we let $\overline{\Gamma} = (x_i : T_i)_k$.*

Lemma 82 1. *(a) $\overline{M^+} = \overline{M}$.* *(b) $\overline{e\Gamma} = e\overline{\Gamma}$*
(c) Let $m \in \mathbb{N}$. If $\overline{M} = N$ and all subterms of M have degree m then M is the unique such term of degree m (i.e., if $\overline{M'} = N$ and all subterms of M' have degree m then $M = M'$).

2. *If $\Gamma = (x_i^{n_i} : T_i)_n$ and $M : \langle \Gamma \vdash_1 T \rangle$, then $\overline{\Gamma} = (x_i : T_i)_n$ is a λI-environment.*

3. *If Γ_1, Γ_2 and $\Gamma_1 \sqcap \Gamma_2$ are $\vdash_1$-legal, then $\overline{\Gamma_1 \sqcap \Gamma_2} = \overline{\Gamma_1} \sqcap \overline{\Gamma_2}$.*

4. *If $M : \langle (x_i : T_i)_n \vdash_v T \rangle$ then $M : \langle (x_i : T_i)_n \vdash_s T \rangle$, T is very good, $\forall\ 1 \le i \le n$, T_i is very good and $d(T_i) = d(T)$.*

5. *If $M : \langle (x_i^{n_i} : T_i)_n \overset{\circ}{\vdash} T \rangle$ then $M : \langle (x_i^{n_i} : T_i)_n \vdash_1 T \rangle$, T is very good, $\forall\ 1 \le i \le n$, T_i is very good, $d(T_i) = n_i = d(T) = d(M)$ and if N is a subterm of M then $d(N) = d(M)$.*

Proof

1 (a) and (c) are by induction on M and (b) is trivial.

2 by lemma 22.3, if $i \neq j$, then $x_i \neq x_j$ and hence $\overline{\Gamma} = (x_i : T_i)_n$ is a λI-environment.

3 Let $\Gamma_1 = (x_i^{n_i} : U_i)_n, (y_j^{m_j} : V_j)_m$ and $\Gamma_2 = (x_i^{n_i} : U'_i)_n, (z_k^{r_k} : W_k)_r$. $\Gamma_1 \sqcap \Gamma_2 = (x_i^{n_i} : U_i \sqcap U'_i)_n, (y_j^{m_j} : V_j)_m, (z_k^{r_k} : W_k)_r$. By 2, $\overline{\Gamma_1} = (x_i : U_i)_n, (y_j : V_j)_m$, $\overline{\Gamma_2} = (x_i : U'_i)_n, (z_k : W_k)_m$ and $\overline{\Gamma_1 \sqcap \Gamma_2} = (x_i : U_i \sqcap U'_i)_n, (y_j : V_j)_m, (z_k : W_k)_r$ are λI-environments and $x_i \neq y_j$ and $x_i \neq z_k$ $\forall i, j, k$.
Hence, $\overline{\Gamma_1} \sqcap \overline{\Gamma_2} = (x_i : U_i \sqcap U'_i)_n, (y_j : V_j)_m, (z_k : W_k)_r = \overline{\Gamma_1 \sqcap \Gamma_2}$.

4. By induction on the derivation of $M : \langle (x_i : T_i)_n \vdash_v T \rangle$.

5. By induction on the derivation of $M : \langle (x_i^{n_i} : T_i)_n \overset{\circ}{\vdash} T \rangle$.

□

The next lemma shows that if indices are removed from a legal typing judgement, then the resulting typing judgement is legal in the λI-calculus using the corresponding intersection type system. The lemma also establishes the result in the other direction for very good types.

Lemma 83 *1. If $M : \langle \Gamma \vdash_1 T\rangle$, then $\overline{M} : \langle \overline{\Gamma} \vdash_s T\rangle$.*

2. If $M : \langle \Gamma \vdash_v T\rangle$, then there are M', Γ' such that $\overline{M'} = M$, $\overline{\Gamma'} = \Gamma$ and $M' : \langle \Gamma' \overset{\circ}{\vdash} T\rangle$. Moreover, such M' and Γ' are unique.

Proof

1. By induction on the derivation of $M : \langle \Gamma \vdash_1 T\rangle$ using lemma 82.
   - ax is trivial.
   - Let $\dfrac{M : \langle \Gamma, (x^m : T_1) \vdash_1 T_2\rangle}{\lambda x^m.M : \langle (x_i^{n_i} : U_i)_n \vdash_1 T_1 \to T_2\rangle}$ where $\Gamma = (x_i^{n_i} : U_i)_n$.
     By lemma 82 and IH, $\overline{\Gamma} = (x_i : U_i)_n$ and $\overline{M} : \langle \overline{\Gamma}, x : T_1 \vdash_s T_2\rangle$.
     Hence, by $\to_i$, $\overline{\lambda x.M} : \langle \overline{\Gamma} \vdash_s T_1 \to T_2\rangle$.
   - $\dfrac{M_1 : \langle \Gamma_1 \vdash_1 T_1 \to T_2\rangle \quad M_2 : \langle \Gamma_2 \vdash_1 T_1\rangle \quad \Gamma_1 \diamond \Gamma_2}{(M_1\ M_2) : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_1 T_2\rangle}$.
     By IH, $\overline{M_1} : \langle \overline{\Gamma_1} \vdash_s T_1 \to T_2\rangle$ and $\overline{M_2} : \langle \overline{\Gamma_2} \vdash_s T_1\rangle$.
     Thus, by $\to_e$ and lemma 82, $\overline{M_1\ M_2} : \langle \overline{\Gamma_1} \sqcap \overline{\Gamma_2} = \overline{\Gamma_1 \sqcap \Gamma_2} \vdash_s T_2\rangle$.
   - Let $\dfrac{M : \langle \Gamma_1 \vdash_1 T_1\rangle \quad M : \langle \Gamma_2 \vdash_1 T_2\rangle}{M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_1 T_1 \sqcap T_2\rangle}$. By IH, $\overline{M} : \langle \overline{\Gamma_1} \vdash_s T_1\rangle$ and $\overline{M} : \langle \overline{\Gamma_2} \vdash_s T_2\rangle$, thus, by $\sqcap_i$ and lemma 82, $\overline{M} : \langle \overline{\Gamma_1} \sqcap \overline{\Gamma_2} = \overline{\Gamma_1 \sqcap \Gamma_2} \vdash_s T_1 \sqcap T_2\rangle$.
   - Let $\dfrac{M : \langle \Gamma \vdash_1 T\rangle}{M^+ : \langle e\Gamma \vdash_1 eT\rangle}$. By IH, *exp* and lemma 82, $\overline{M} : \langle e\overline{\Gamma} = \overline{e\Gamma} \vdash_s eT\rangle$.

2. First, we prove the existence of M' and Γ' by induction on the derivation of $M : \langle \Gamma \vdash_v T\rangle$ using lemma 82.
   - ax_v is trivial.
   - Let $\dfrac{M : \langle \Gamma, (x : T_1) \vdash_v T_2\rangle}{\lambda x.M : \langle \Gamma \vdash_v T_1 \to T_2\rangle}$. By IH, there are M', Γ' such that $\overline{\Gamma'} = \Gamma$, $\overline{M'} = M$, and $M' : \langle \Gamma', x^m : T_1 \overset{\circ}{\vdash} T_2\rangle$.
     Hence, by $\to_i$, $\lambda x^m.M' : \langle \Gamma \overset{\circ}{\vdash} T_1 \to T_2\rangle$.
   - Let $\dfrac{M_1 : \langle \Gamma_1 \vdash_v T_1 \to T_2\rangle \quad M_2 : \langle \Gamma_2 \vdash_v T_1\rangle}{(M_1\ M_2) : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_v T_2\rangle}$. By IH, there are M_1', M_2', Γ_1', Γ_2', such that $\overline{M_1'} = M_1$, $\overline{M_2'} = M_2$, $\overline{\Gamma_1'} = \Gamma_1$, $\overline{\Gamma_2'} = \Gamma_2$, $M_1' : \langle \Gamma_1' \overset{\circ}{\vdash} T_1 \to T_2\rangle$ and $M_2' : \langle \Gamma_2' \overset{\circ}{\vdash} T_1\rangle$. By lemma 82.(4 and 5), $T_1 \to T_2$ is very good (hence $\mathrm{d}(T_1) = \mathrm{d}(T_2)$) and if $x^n \in FV(M_1')$, $x^m \in FV(M_2')$ then $\mathrm{d}(x^n) = n = \mathrm{d}(M_1')$ and $\mathrm{d}(x^m) = m = \mathrm{d}(M_2')$. By lemma 82.5 and lemma 23, $\mathrm{d}(M_1') = \mathrm{d}(T_1 \to T_2) = \mathrm{d}(T_2)$ and $\mathrm{d}(M_2') = \mathrm{d}(T_2)$. Hence, $m = n$. By lemma 22.4, $\Gamma_1' \diamond \Gamma_2'$, Hence, by $\to_e$, $M_1'M_2' : \langle \Gamma_1' \sqcap \Gamma_2' \overset{\circ}{\vdash} T_2\rangle$.
   - Let $\dfrac{M : \langle \Gamma_1 \vdash_v T_1\rangle \quad M : \langle \Gamma_2 \vdash_v T_2\rangle \quad \mathrm{d}(T_1) = \mathrm{d}(T_2)}{M : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_v T_1 \sqcap T_2\rangle}$.
     By IH, there are $M', M'', \Gamma_1', \Gamma_2'$ such that $\overline{M'} = \overline{M''} = M$, $\overline{\Gamma_1'} = \Gamma_1$, $\overline{\Gamma_2'} = \Gamma_2$, $M' : \langle \Gamma_1' \overset{\circ}{\vdash} T_1\rangle$, and $M'' : \langle \Gamma_2' \overset{\circ}{\vdash} T_2\rangle$. By lemma 82.(1 and 5), $M' = M''$ and hence by $\sqcap_i$, $M' : \langle \Gamma_1' \sqcap \Gamma_2' \overset{\circ}{\vdash} T_1 \sqcap T_2\rangle$.

- Let $\dfrac{M : \langle \Gamma \vdash_v T \rangle}{M^+ : \langle e\Gamma \vdash_v eT \rangle}$. Use IH, *exp* and lemma 82.

Now we prove unicity of M' and Γ'. Assume that there are M'', Γ'' such that $\overline{M''} = M$, $\overline{\Gamma''} = \Gamma$ and $M'' : \langle \Gamma'' \overset{\circ}{\vdash} T \rangle$. By lemma 82.(1 and 5) and lemma 23, $M' = M''$. Moreover, since $\overline{\Gamma'} = \overline{\Gamma''} = \Gamma$ then let $\Gamma' = (x_i^{n_i} : U_i)_n$, $\Gamma'' = (x_i^{m_i} : U_i)_n$ and $\Gamma = (x_i : U_i)_n$. By lemma 82.5, $\forall 1 \leq i \leq n$, $m_1 = n_i = \mathrm{d}(T)$. Hence, $\Gamma' = \Gamma''$.

$\square$

E Removing indices from $\vdash_2$

In this section we show that our results for $\vdash_2$ can be translated to the λI-calculus (i.e., where indices are removed). We use the notations of section D.

Definition 84 *1. The typing rules of the λI-calculus are given as follows:*

$$\frac{}{x : \langle (x : T) \vdash_2' T \rangle}\ ax$$

$$\frac{M : \langle \Gamma, (x : U) \vdash_2' T \rangle}{\lambda x.M : \langle \Gamma \vdash_2' U \to T \rangle}\ \to_i$$

$$\frac{M_1 : \langle \Gamma_1 \vdash_2' U \to T \rangle \qquad M_2 : \langle \Gamma_2 \vdash_2' U \rangle}{M_1 M_2 : \langle \Gamma_1 \sqcap \Gamma_2 \vdash_2' T \rangle}\ \to_e$$

$$\frac{M : \langle \Gamma \vdash_2' U_1 \rangle \qquad M : \langle \Gamma \vdash_2' U_2 \rangle}{M : \langle \Gamma \vdash_2' U_1 \sqcap U_2 \rangle}\ \sqcap_i$$

$$\frac{M : \langle \Gamma \vdash_2' U \rangle}{M : \langle e\Gamma \vdash_2' eU \rangle}\ exp$$

$$\frac{M : \langle \Gamma \vdash_2' U \rangle \qquad \langle \Gamma \vdash_2' U \rangle \sqsubseteq' \langle \Gamma' \vdash_2' U' \rangle}{M : \langle \Gamma' \vdash_2' U' \rangle}\ \sqsubseteq'$$

In the last clause, the binary relation $\sqsubseteq'$ is defined by the following rules:

$$\frac{}{\Phi \sqsubseteq' \Phi}\ ref$$

$$\frac{\Phi_1 \sqsubseteq' \Phi_2 \qquad \Phi_2 \sqsubseteq' \Phi_3}{\Phi_1 \sqsubseteq' \Phi_3}\ tr$$

$$\frac{}{U_1 \sqcap U_2 \sqsubseteq' U_1}\ \sqcap_e$$

$$\frac{U_1 \sqsubseteq' V_1 \quad \& \quad U_2 \sqsubseteq' V_2}{U_1 \sqcap U_2 \sqsubseteq' V_1 \sqcap V_2}\ \sqcap$$

$$\frac{U_2 \sqsubseteq' U_1 \quad \& \quad T_1 \sqsubseteq' T_2}{U_1 \to T_1 \sqsubseteq' U_2 \to T_2}\ \to$$

$$\frac{U_1 \sqsubseteq' U_2}{eU_1 \sqsubseteq' eU_2}\ \sqsubseteq'_{exp}$$

$$\frac{U_1 \sqsubseteq' U_2}{\Gamma, (y : U_1) \sqsubseteq' \Gamma, (y : U_2)} \quad \sqsubseteq'_c$$

$$\frac{U_1 \sqsubseteq' U_2 \quad \& \quad \Gamma_2 \sqsubseteq' \Gamma_1}{\langle \Gamma_1 \vdash'_2 U_1 \rangle \sqsubseteq' \langle \Gamma_2 \vdash'_2 U_2 \rangle} \quad \sqsubseteq'_{\langle\rangle}$$

2. *We define* $\overline{M}$ *and* $\overline{\Gamma}$ *as in definition 81.*
If $\langle \Gamma \vdash_2 U \rangle$ *is a typing, then we let* $\overline{\langle \Gamma \vdash_2 U \rangle} = \langle \overline{\Gamma} \vdash'_2 U \rangle$.

Lemma 85 1. *If* $U \sqsubseteq U'$ *then* $U \sqsubseteq' U'$.

2. *If* $\Gamma \sqsubseteq' \Gamma'$, $U \sqsubseteq' U'$ *and* $x \notin dom(\Gamma)$ *then* $\Gamma, (x : U) \sqsubseteq' \Gamma', (x : U')$.

3. $\Gamma \sqsubseteq' \Gamma'$ *iff* $\Gamma = (x_i : U_i)_n$, $\Gamma' = (x_i : U'_i)_n$ *and for every* $1 \leq i \leq n$, $U_i \sqsubseteq' U'_i$.

4. $\langle \Gamma \vdash_2 U \rangle \sqsubseteq' \langle \Gamma' \vdash_2 U' \rangle$ *iff* $\Gamma' \sqsubseteq' \Gamma$ *and* $U \sqsubseteq' U'$.

Proof 1. By induction on $U \sqsubseteq U'$. The rest is similar to the proof of lemma 21. □

Lemma 86 1. *(a)* $\overline{M^+} = \overline{M}$. *(b)* $\overline{e\Gamma} = e\overline{\Gamma}$.

2. *If* $\Gamma = (x_i^{n_i} : U_i)_n$ *and* $M : \langle \Gamma \vdash_2 U \rangle$, *then* $\overline{\Gamma} = (x_i : U_i)_n$ *is a* λI*-environment.*

3. *If* Γ_1, Γ_2 *and* $\Gamma_1 \sqcap \Gamma_2$ *are* $\vdash_2$*-legal, then* $\overline{\Gamma_1 \sqcap \Gamma_2} = \overline{\Gamma_1} \sqcap \overline{\Gamma_2}$.

4. *If* $\Gamma \sqsubseteq \Gamma'$, *then* $\overline{\Gamma} \sqsubseteq' \overline{\Gamma'}$.

5. *If* $\langle \Gamma \vdash_2 U \rangle \sqsubseteq \langle \Gamma' \vdash_2 U' \rangle$, *then* $\overline{\langle \Gamma \vdash_2 U \rangle} \sqsubseteq' \overline{\langle \Gamma' \vdash_2 U' \rangle}$.

Proof For 1, 2, 3, see lemma 82. 4. use lemmas 21.2, 85.1 and 85.3.
5. use lemmas 21.3, 4, 85.1 and 85.4. □

Lemma 87 *If* $M : \langle \Gamma \vdash_2 U \rangle$, *then* $\overline{M} : \langle \overline{\Gamma} \vdash'_2 U \rangle$

Proof By induction on the derivation of $M : \langle \Gamma \vdash_2 U \rangle$ using lemma 86.3 in $\to_e$ and lemma 86.5 in $\sqsubseteq'$. We examine the rule $\sqsubseteq'$.

Let $\dfrac{M : \langle \Gamma \vdash_2 U \rangle \qquad \langle \Gamma \vdash_2 U \rangle \sqsubseteq \langle \Gamma' \vdash_2 U' \rangle}{M : \langle \Gamma' \vdash_2 U' \rangle}$. By IH, $\overline{M} : \langle \overline{\Gamma} \vdash'_2 U \rangle$, and, by lemma 86.5, $\langle \overline{\Gamma} \vdash'_2 U \rangle \sqsubseteq' \langle \overline{\Gamma'} \vdash'_2 U' \rangle$, then $\overline{M} : \langle \overline{\Gamma'} \vdash'_2 U' \rangle$. □