Bar Induction: The Good, the Bad and the Ugly

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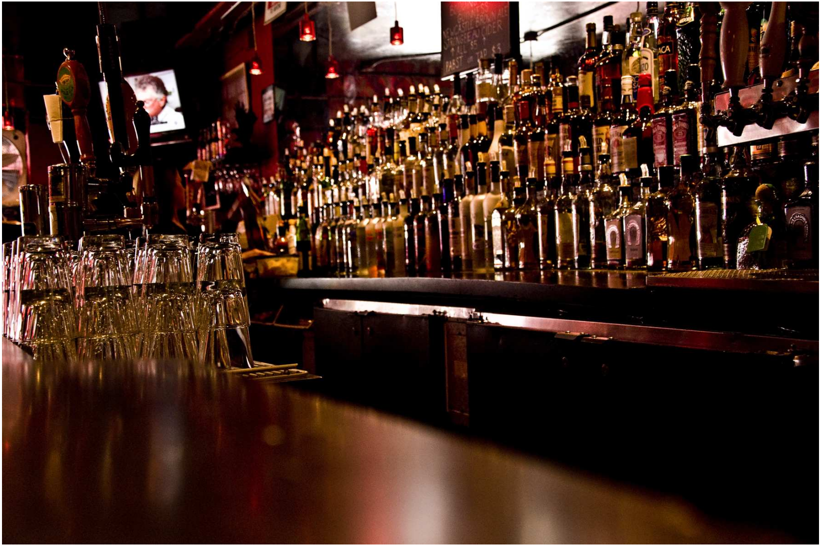
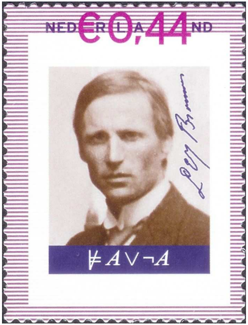
Bar induction?

What bar induction is **not** about?



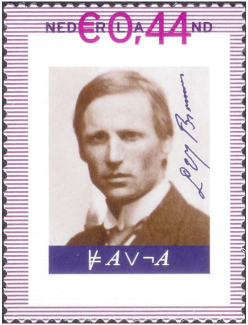
(source: https://get.taphunter.com/blog/4-ways-to-ensure-your-bar-rocks/)

Intuitionism



- **First act**: Intuitionistic logic is based on our **inner consciousness of time**, which gives rise to the **two-ity**.
- As opposed to Platonism, it's about **constructions in the mind** and not objects that exist independently of us. There are no mathematical truths outside human thought: "all mathematical truths are experienced truths" (Brouwer)
- A statement is true when we have an appropriate construction, and false when no construction is possible.

Intuitionism



- **Second act**: New mathematical entities can be created through **more or less freely proceeding sequences** of mathematical entities.
- Also by defining new mathematical species (types, sets) that respect equality of mathematical entities.
- Gives rise to (never finished) choice sequences. Could be lawlike or lawless. Laws can be 1st order, 2nd order...
- The continuum is captured by choice sequences of nested rational intervals.

Intuitionism—Continuity

What can we do with these never finished sequences?

Brouwer's answer: one never needs the whole sequence.

His **continuity axiom for numbers** says that functions from sequences to numbers only need initial segments

$$\forall F : \mathbb{N}^{\mathcal{B}}.\ \forall \alpha : \mathcal{B}.\ \exists n : \mathbb{N}.\ \forall \beta : \mathcal{B}.\ \alpha =_{\mathcal{B}_n} \beta \rightarrow F(\alpha) =_{\mathbb{N}} F(\beta)$$

From which his **uniform continuity theorem** follows: Let f be of type $[\alpha, \beta] \rightarrow \mathbb{R}$, then

$$\forall \epsilon > 0. \exists \delta > 0. \forall x, y : [\alpha, \beta].\ |x - y| \leqslant \delta \rightarrow |f(x) - f(y)| \leqslant \epsilon$$

$(\mathcal{B} = \mathbb{N}^{\mathbb{N}}\ \&\ \mathcal{B}_n = \mathbb{N}^{\mathbb{N}_n})$

Intuitionism—Continuity

False (Kreisel 62, Troelstra 77, Escardó & Xu 2015):

$$\neg \mathbf{\Pi} F{:}\mathbb{N}^{\mathcal{B}}.\mathbf{\Pi}\alpha{:}\mathcal{B}.\mathbf{\Sigma} n{:}\mathbb{N}.\mathbf{\Pi}\beta{:}\mathcal{B}.\alpha =_{\mathcal{B}_n} \beta \to F(\alpha) =_{\mathbb{N}} F(\beta)$$

(no continuous way of finding a modulus of continuity of a given function F at a point α)

($\exists F = G : \mathbb{N}^{\mathcal{B}}$. F and G have different moduli of continuity)

True in Nuprl (see our CPP 2016 paper):

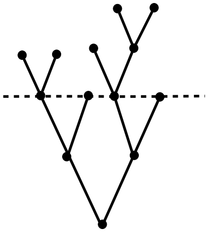
$$\mathbf{\Pi} F{:}\mathcal{B} \to \mathbb{N}.\mathbf{\Pi}\alpha{:}\mathcal{B}.\downarrow\mathbf{\Sigma} n{:}\mathbb{N}.\mathbf{\Pi}\beta{:}\mathcal{B}.\alpha =_{\mathcal{B}_n} \beta \to F(\alpha) =_{\mathbb{N}} F(\beta)$$

Intuitionism—Bar induction

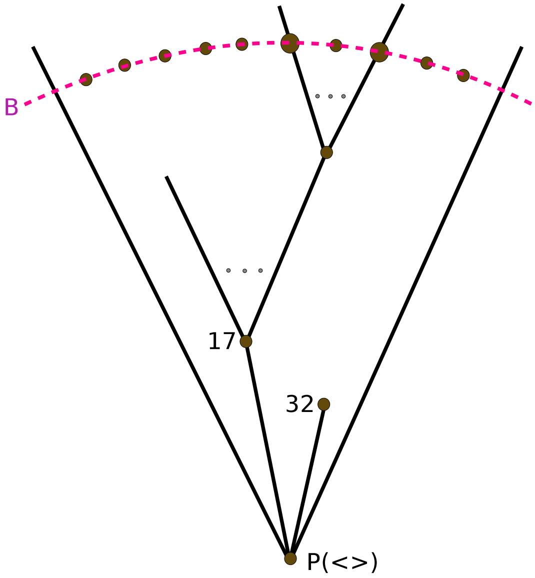
To prove his **uniform continuity theorem**, Brouwer also used the **Fan theorem**.

Which follows from **bar induction**.

The fan theorem says that if for each branch α of a binary tree T, a property A is true about some initial segment of α, then **there is a uniform bound** on the depth at which A is met.



Bar Induction—The intuition



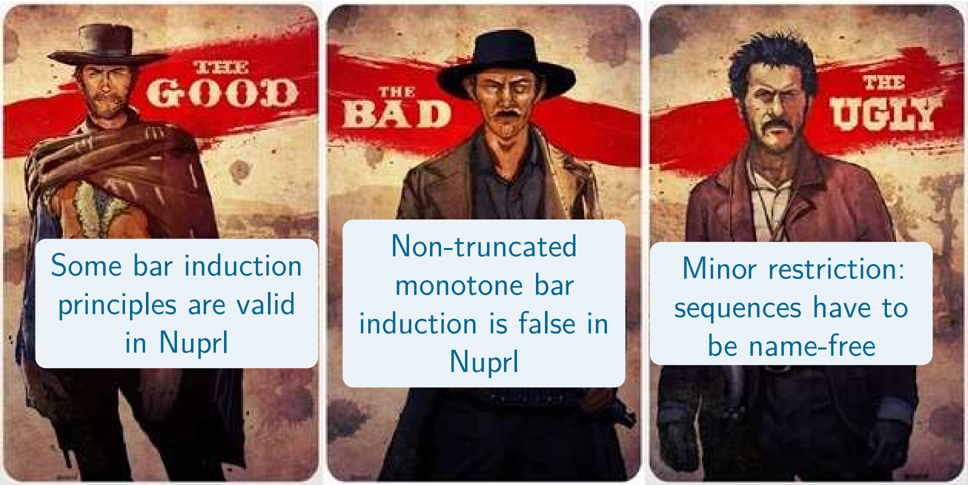
What is this talk about?

What this talk is not about

Not about the philosophical foundations of intuitionism
Not about which foundation is best
About useful constructions

(source: https://sententiaeantiquae.com/2014/10/23)

What is this talk about?



Some bar induction principles are valid in Nuprl

Non-truncated monotone bar induction is false in Nuprl

Minor restriction: sequences have to be name-free

(source: http://cinetropolis.net/scene-is-believing-the-good-the-bad-and-the-ugly/)

Nuprl?

Nuprl in a Nutshell

Became operational in 1984 (Constable & Bates)

Similar to Coq and Agda

Extensional Constructive Type Theory with partial functions

Types are interpreted as
Partial Equivalence Relations on terms (PERs)

Consistency proof in Coq (see our ITP 2014):
https://github.com/vrahli/NuprlInCoq

Extensional CTT with partial functions?

Extensional

$$(\forall a : A.\ f(a) = g(a) \in B) \rightarrow f = g \in A \rightarrow B$$

Constructive

$(A \rightarrow A)$ true because inhabited by $(\lambda x.x)$

Partial functions

$\mathtt{fix}(\lambda x.x)$ inhabits $\overline{\mathbb{N}}$

Nuprl Types—Martin-Löf's extensional type theory

Equality

$$a = b \in T$$

Dependent product

$$a{:}A \to B[a] \qquad \text{or} \qquad \mathbf{\Pi} a{:}A.B[a]$$

Dependent sum

$$a{:}A \times B[a] \qquad \text{or} \qquad \mathbf{\Sigma} a{:}A.B[a]$$

Universe

$$\mathbb{U}_i$$

Nuprl Types—Less “conventional types”

Partial: $\overline{A}$

Disjoint union: $A+B$

Intersection: $\cap a{:}A.B[a]$

Union: $\cup a{:}A.B[a]$

Set: $\{a : A \mid B[a]\}$

Quotient: $T//E$

Domain: `Base`

Simulation: $t_1 \preccurlyeq t_2$

($\texttt{Void} = 0 \preccurlyeq 1$ and $\texttt{Unit} = 0 \preccurlyeq 0$)

Bisimulation: $t_1 \sim t_2$

Image: $\texttt{Img}(A, f)$

PER: $\texttt{per}(R)$

Nuprl Types—Squashing/Truncation

$\{\texttt{Unit} \mid T\}$

Proof erasure: $\downarrow T$

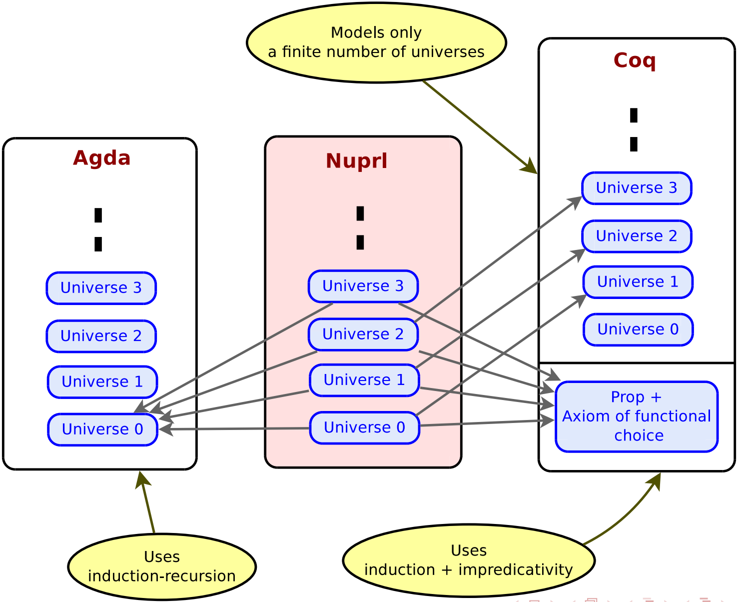
$\texttt{Img}(T, \lambda\_.\star)$

Proof irrelevance: $\downarrow T$ $T//\texttt{True}$

For example:

$\mathbf{\Pi} P{:}\mathbb{P}.(P \vee \neg P)$	✗
$\mathbf{\Pi} P{:}\mathbb{P}.\downarrow(P \vee \neg P)$	✗
$\mathbf{\Pi} P{:}\mathbb{P}.\downarrow(P \vee \neg P)$	✓

Nuprl PER Semantics Implemented in Coq



Bar Induction in Nuprl

Bar Induction—Non-intuitionistic

in Coq

$H \vdash \downarrow P(0, \perp\!\!\!\perp)$

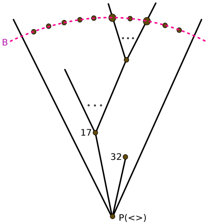
BY [BID]

(wfd) $\quad H, n : \mathbb{N}, s : \mathcal{B}_n \vdash B(n, s) \in \texttt{Type}$

(bar) $\quad H, s : \mathcal{B} \vdash \downarrow \exists n : \mathbb{N}.\ B(n, s)$

(imp) $\quad H, n : \mathbb{N}, s : \mathcal{B}_n, m : B(n, s) \vdash P(n, s)$

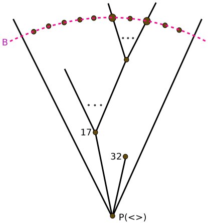
(ind) $\quad H, n : \mathbb{N}, s : \mathcal{B}_n, x : (\forall m : \mathbb{N}.\ P((n + 1), s \oplus_n m)) \vdash P(n, s)$



Bar Induction—On decidable bars

in Nuprl

$$
\begin{array}{ll}
\multicolumn{2}{l}{H \vdash P(0, \perp\!\!\!\perp)} \\
\quad \text{BY [BID]} & \\
\quad (\mathtt{dec}) & H, n : \mathbb{N}, s : \mathcal{B}_n \vdash B(n, s) \;\vee\; \neg B(n, s) \\
\quad (\mathtt{bar}) & H, s : \mathcal{B} \vdash \downarrow\exists n : \mathbb{N}.\ B(n, s) \\
\quad (\mathtt{imp}) & H, n : \mathbb{N}, s : \mathcal{B}_n, m : B(n, s) \vdash P(n, s) \\
\quad (\mathtt{ind}) & H, n : \mathbb{N}, s : \mathcal{B}_n, x : (\forall m : \mathbb{N}.\ P((n+1), s \oplus_n m)) \vdash P(n, s)
\end{array}
$$



Bar Induction—On monotone bars

in Nuprl

$H \vdash \downharpoonleft P(0, \perp\!\!\!\perp)$

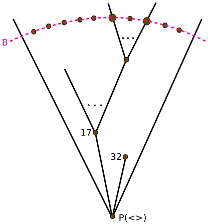
BY [BIM]

(mon) $H, n : \mathbb{N}, s : \mathcal{B}_n \vdash \forall m : \mathbb{N}.\ B(n, s) \Rightarrow B(n+1, s \oplus_n m)$

(bar) $H, s : \mathcal{B} \vdash \downharpoonleft \exists n : \mathbb{N}.\ B(n, s)$

(imp) $H, n : \mathbb{N}, s : \mathcal{B}_n, m : B(n, s) \vdash P(n, s)$

(ind) $H, n : \mathbb{N}, s : \mathcal{B}_n, x : (\forall m : \mathbb{N}.\ P((n+1), s \oplus_n m)) \vdash P(n, s)$



Bar Induction—Why the squashing operator?

Continuity is false in Martin-Löf-like type theories
when not $\downharpoonleft$-squashed

$$(A)\ \mathbf{\Pi} F{:}\mathbb{N}^{\mathcal{B}}.\mathbf{\Pi} f{:}\mathcal{B}.\downharpoonleft \mathbf{\Sigma} n{:}\mathbb{N}.\mathbf{\Pi} g{:}\mathcal{B}.f =_{\mathcal{B}_n} g \rightarrow F(f) =_{\mathbb{N}} F(g)$$

$$(B)\ \neg \mathbf{\Pi} F{:}\mathbb{N}^{\mathcal{B}}.\mathbf{\Pi} f{:}\mathcal{B}.\mathbf{\Sigma} n{:}\mathbb{N}.\mathbf{\Pi} g{:}\mathcal{B}.f =_{\mathcal{B}_n} g \rightarrow F(f) =_{\mathbb{N}} F(g)$$

From which we derived:
BIM is false when not $\downharpoonleft$-squashed

otherwise we could derive

$$\mathbf{\Pi} F{:}\mathbb{N}^{\mathcal{B}}.\mathbf{\Pi} f{:}\mathcal{B}.\mathbf{\Sigma} n{:}\mathbb{N}.\mathbf{\Pi} g{:}\mathcal{B}.f =_{\mathcal{B}_n} g \rightarrow F(f) =_{\mathbb{N}} F(g)$$

from BIM & (A)

Bar Induction—Sequences of numbers

We derived BID/BIM for sequences of numbers (easy)

We added "choice sequences" of numbers to Nuprl's model:
all Coq functions from $\mathbb{N}$ to $\mathbb{N}$

What about sequences of terms?

Bar Induction—Sequences of terms

We derived BID for sequences of closed name-free terms

Harder because we turned our terms into a big W type:
Coq functions from $\mathbb{N}$ to terms are now terms!

Why without names?

ν picks fresh names and we can't compute
the collection of all names anymore

Bar Induction—Questions

Can we prove continuity for sequences of terms instead of $\mathcal{B}$?

What does that give us? $+$ proof-theoretic strength?

Can we hope to prove BID/BIM in Coq without LEM/AC?

We're working on this:

Can we derive BID/BIM for sequences of terms with names?

Name	Formula	Where	Comments
$\mathtt{WCP}_{1,0}$	$\neg\mathbf{\Pi}F{:}\mathbb{N}^{\mathcal{B}}.\mathbf{\Pi}f{:}\mathcal{B}.\mathbf{\Sigma}n{:}\mathbb{N}.\mathbf{\Pi}g{:}\mathcal{B}.f =_{\mathcal{B}_n} g \rightarrow F(f) =_{\mathbb{N}} F(g)$	Nuprl	
$\mathtt{WCP}_{1,0_{\downharpoonleft}}$	$\mathbf{\Pi}F{:}\mathbb{N}^{\mathcal{B}}.\mathbf{\Pi}f{:}\mathcal{B}.\downharpoonleft\mathbf{\Sigma}n{:}\mathbb{N}.\ \mathbf{\Pi}g{:}\mathcal{B}.f =_{\mathcal{B}_n} g \rightarrow F(f) =_{\mathbb{N}} F(g)$	Coq	uses named exceptions
$\mathtt{WCP}_{1,0_{\downarrow}}$	$\mathbf{\Pi}F{:}\mathbb{N}^{\mathcal{B}}.\mathbf{\Pi}f{:}\mathcal{B}.\downarrow\mathbf{\Sigma}n{:}\mathbb{N}.\mathbf{\Pi}g{:}\mathcal{B}.f =_{\mathcal{B}_n} g \rightarrow F(f) =_{\mathbb{N}} F(g)$	Coq	uses $\perp$
$\mathtt{WCP}_{1,1}$	$\neg\mathbf{\Pi}P{:}\mathcal{B} \rightarrow \mathbb{P}^{\mathcal{B}}.(\mathbf{\Pi}a{:}\mathcal{B}.\mathbf{\Sigma}b{:}\mathcal{B}.P(a,b)) \rightarrow \mathbf{\Sigma}c{:}\mathbb{N}^{\mathcal{B}}.\mathtt{CONT}(c) \ \wedge\ \mathbf{\Pi}a{:}\mathcal{B}.\mathtt{shift}(c,a)$	Nuprl	
$\mathtt{WCP}_{1,1_{\downharpoonleft}}$	$?\neg\mathbf{\Pi}P{:}\mathcal{B} \rightarrow \mathbb{P}^{\mathcal{B}}.(\mathbf{\Pi}a{:}\mathcal{B}.\mathbf{\Sigma}b{:}\mathcal{B}.P(a,b)) \rightarrow \downharpoonleft\mathbf{\Sigma}c{:}\mathbb{N}^{\mathcal{B}}.\ \mathtt{CONT}(c)_{\downharpoonleft} \ \wedge\ \mathbf{\Pi}a{:}\mathcal{B}.\mathtt{shift}(c,a)$	?	
$\mathtt{WCP}_{1,1_{\downarrow}}$	$?\neg\mathbf{\Pi}P{:}\mathcal{B} \rightarrow \mathbb{P}^{\mathcal{B}}.(\mathbf{\Pi}a{:}\mathcal{B}.\mathbf{\Sigma}b{:}\mathcal{B}.P(a,b)) \rightarrow \downarrow\mathbf{\Sigma}c{:}\mathbb{N}^{\mathcal{B}}.\mathtt{CONT}(c)_{\downarrow} \ \wedge\ \mathbf{\Pi}a{:}\mathcal{B}.\mathtt{shift}(c,a)$	?	
$\mathtt{AC}_{0,0}$	$\mathbf{\Pi}P{:}\mathbb{N} \rightarrow \mathbb{P}^{\mathbb{N}}.(\mathbf{\Pi}n{:}\mathbb{N}.\mathbf{\Sigma}m{:}\mathbb{N}.P(n,m)) \rightarrow \mathbf{\Sigma}f{:}\mathcal{B}.\mathbf{\Pi}n{:}\mathcal{B}.P(n,f(n))$	Nuprl	
$\mathtt{AC}_{0,0_{\downharpoonleft}}$	$\mathbf{\Pi}P{:}\mathbb{N} \rightarrow \mathbb{P}^{\mathbb{N}}.(\mathbf{\Pi}n{:}\mathbb{N}.\downharpoonleft\mathbf{\Sigma}m{:}\mathbb{N}.\ P(n,m)) \rightarrow \downharpoonleft\mathbf{\Sigma}f{:}\mathcal{B}.\ \mathbf{\Pi}n{:}\mathcal{B}.P(n,f(n))$	Nuprl	
$\mathtt{AC}_{0,0_{\downarrow}}$	$\mathbf{\Pi}P{:}\mathbb{N} \rightarrow \mathbb{P}^{\mathbb{N}}.(\mathbf{\Pi}n{:}\mathbb{N}.\downarrow\mathbf{\Sigma}m{:}\mathbb{N}.P(n,m)) \rightarrow \downarrow\mathbf{\Sigma}f{:}\mathcal{B}.\mathbf{\Pi}n{:}\mathcal{B}.P(n,f(n))$	Coq	uses classical logic
$\mathtt{AC}_{1,0}$	$\mathbf{\Pi}P{:}\mathcal{B} \rightarrow \mathbb{P}^{\mathbb{N}}.(\mathbf{\Pi}f{:}\mathcal{B}.\mathbf{\Sigma}n{:}\mathbb{N}.P(f,n)) \rightarrow \mathbf{\Sigma}F{:}\mathbb{N}^{\mathcal{B}}.\mathbf{\Pi}f{:}\mathcal{B}.P(f,F(f))$	Nuprl	
$\mathtt{AC}_{1,0_{\downharpoonleft}}$	$\mathbf{\Pi}P{:}\mathcal{B} \rightarrow \mathbb{P}^{\mathbb{N}}.(\mathbf{\Pi}f{:}\mathcal{B}.\downharpoonleft\mathbf{\Sigma}n{:}\mathbb{N}.\ P(f,n)) \rightarrow \downharpoonleft\mathbf{\Sigma}F{:}\mathbb{N}^{\mathcal{B}}.\ \mathbf{\Pi}f{:}\mathcal{B}.P(f,F(f))$	Nuprl	
$\mathtt{AC}_{1,0_{\downarrow}}$	$?\neg\mathbf{\Pi}P{:}\mathcal{B} \rightarrow \mathbb{P}^{\mathbb{N}}.(\mathbf{\Pi}f{:}\mathcal{B}.\downarrow\mathbf{\Sigma}n{:}\mathbb{N}.P(f,n)) \rightarrow \downarrow\mathbf{\Sigma}F{:}\mathbb{N}^{\mathcal{B}}.\mathbf{\Pi}f{:}\mathcal{B}.P(f,F(f))$	?	
$\mathtt{AC}_{2,0}$	$\mathbf{\Pi}P{:}\mathbb{N}^{\mathcal{B}} \rightarrow \mathbb{P}^{\mathbb{N}}.(\mathbf{\Pi}f{:}\mathbb{N}^{\mathcal{B}}.\mathbf{\Sigma}n{:}T.P(f,n)) \rightarrow \mathbf{\Sigma}F{:}T^{(\mathbb{N}^{\mathcal{B}})}.\mathbf{\Pi}f{:}\mathbb{N}^{\mathcal{B}}.P(f,F(f))$	Nuprl	
$\mathtt{AC}_{2,0_{\downharpoonleft}}$	$\neg(\mathbf{\Pi}P{:}\mathbb{N}^{\mathcal{B}} \rightarrow \mathbb{P}^{T}.(\mathbf{\Pi}f{:}\mathbb{N}^{\mathcal{B}}.\downharpoonleft\mathbf{\Sigma}n{:}T.\ P(f,n)) \rightarrow \downharpoonleft\mathbf{\Sigma}F{:}T^{(\mathbb{N}^{\mathcal{B}})}.\ \mathbf{\Pi}f{:}\mathbb{N}^{\mathcal{B}}.P(f,F(f)))$	Nuprl	contradicts continuity
$\mathtt{AC}_{2,0_{\downarrow}}$	$\neg(\mathbf{\Pi}P{:}\mathbb{N}^{\mathcal{B}} \rightarrow \mathbb{P}^{T}.(\mathbf{\Pi}f{:}\mathbb{N}^{\mathcal{B}}.\downharpoonleft\mathbf{\Sigma}n{:}T.\ P(f,n)) \rightarrow \downarrow\mathbf{\Sigma}F{:}T^{(\mathbb{N}^{\mathcal{B}})}.\mathbf{\Pi}f{:}\mathbb{N}^{\mathcal{B}}.P(f,F(f)))$	Nuprl	contradicts continuity
LEM	$\neg\mathbf{\Pi}P{:}\mathbb{P}.P \ \vee\ \neg P$	Nuprl	
$\text{LEM}_{\downharpoonleft}$	$\neg\mathbf{\Pi}P{:}\mathbb{P}.\downharpoonleft(P \ \vee\ \neg P)$	Nuprl	
$\text{LEM}_{\downarrow}$	$\mathbf{\Pi}P{:}\mathbb{P}.\downarrow(P \ \vee\ \neg P)$	Coq	uses classical logic
MP	$\mathbf{\Pi}P{:}\mathbb{P}^{\mathbb{N}}.(\mathbf{\Pi}n{:}\mathbb{N}.P(n) \ \vee\ \neg P(n)) \rightarrow (\neg\mathbf{\Pi}n{:}\mathbb{N}.\neg P(n)) \rightarrow \mathbf{\Sigma}n{:}\mathbb{N}.P(n)$	Nuprl	uses $\text{LEM}_{\downarrow}$
KS	$\neg\mathbf{\Pi}A{:}\mathbb{P}.\mathbf{\Sigma}a{:}\mathcal{B}.((\mathbf{\Sigma}x{:}\mathbb{N}.a(x) =_{\mathbb{N}} 1) \iff A)$	Nuprl	uses MP
$\text{KS}_{\downharpoonleft}$	$\neg\mathbf{\Pi}A{:}\mathbb{P}.\downharpoonleft\mathbf{\Sigma}a{:}\mathcal{B}.((\mathbf{\Sigma}x{:}\mathbb{N}.a(x) =_{\mathbb{N}} 1) \iff A)$	Nuprl	uses MP
$\text{KS}_{\downarrow}$	$\mathbf{\Pi}A{:}\mathbb{P}.\downarrow\mathbf{\Sigma}a{:}\mathcal{B}.((\mathbf{\Sigma}x{:}\mathbb{N}.a(x) =_{\mathbb{N}} 1) \iff A)$	Coq	uses classical logic
$\text{BI}_{\downarrow}$	$\mathtt{WF}(B) \rightarrow \mathtt{BAR}_{\downharpoonleft}(B) \rightarrow \mathtt{BASE}(B,P) \rightarrow \mathtt{IND}(P) \rightarrow \downarrow P(0,\perp\!\!\!\perp)$	Coq	uses classical logic
BID	$\mathtt{WF}(B) \rightarrow \mathtt{BAR}_{\downharpoonleft}(B) \rightarrow \mathtt{DEC}(B) \rightarrow \mathtt{BASE}(B,P) \rightarrow \mathtt{IND}(P) \rightarrow P(0,\perp\!\!\!\perp)$	Nuprl	uses $\text{BI}_{\downarrow}$
$\text{BIM}_{\downharpoonleft}$	$\mathtt{WF}(B) \rightarrow \mathtt{BAR}_{\downharpoonleft}(B) \rightarrow \mathtt{MON}(B) \rightarrow \mathtt{BASE}(B,P) \rightarrow \mathtt{IND}(P) \rightarrow \downharpoonleft P(0,\perp\!\!\!\perp)$	Nuprl	uses $\text{BI}_{\downarrow}$
BIM	$\neg\mathbf{\Pi}B,P{:}(\mathbf{\Pi}n{:}\mathbb{N}.\mathbb{P}^{\mathcal{B}_n}).\mathtt{BAR}_{\downharpoonleft}(B) \rightarrow \mathtt{MON}(B) \rightarrow \mathtt{BASE}(B,P) \rightarrow \mathtt{IND}(P) \rightarrow P(0,\perp\!\!\!\perp)$	Nuprl	contradicts continuity