Reducibility proofs in λ-calculi with intersection types

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Interest

- By using reducibility, new, simple and general methods can be developed to prove properties of the λ-calculus.
- In our paper:
  - We review and find the flaws in one reducibility method of proofs of Church-Rosser, standardisation and weak head normalisation.
  - We review, adapt and non trivially extend another reducibility method of proofs of Church-Rosser.

The Two Reducibility Methods

1. Ghilezan and Likavec's method:
   ➤ According to this method, a certain property of the λ-calculus is proved to hold, if that property satisfies a certain set of predicates.
   ➤ Unfortunately, this method does not work. We give counterexamples.
2. Koletsos and Stavrinos's method:
   ➤ This method aims to prove the Church-Rosser property of the untyped λ-calculus by showing first that a typed λ-calculus is confluent and using this to show the confluence of developments.
   ➤ We adapt this method to βI-reduction.
   ➤ We extend (this is non trivial) this method to $\beta\eta$-reduction.

Ghilezan and Likavec's Method [GL02]

Ghilezan and Likavec designed a general proof method schema.

The basic step of the method: if a set of λ-terms $\mathcal{P}$ satisfies a defined set of predicates pred then it contains a certain set of typable λ-terms T.

- $\mathrm{pred}(\mathcal{P}) \Rightarrow T \subseteq \mathcal{P}$

Extension of the basic step: if a set of λ-terms $\mathcal{P}$ satisfies a defined set of predicates pred then it contains the whole set of λ-terms.

- $\mathrm{pred}(\mathcal{P}) \Rightarrow \Lambda = \mathcal{P}$

Ghilezan and Likavec's method [GL02]

the basic step in a simple framework

Below, $\mathcal{P}$ is a set of terms. Using:

- a set of types $\sigma \in \mathsf{Type}^1 ::= \alpha \mid \sigma_1 \to \sigma_2 \mid \sigma_1 \cap \sigma_2$,
- a type interpretation function $[\![-]\!]^1_{\mathcal{P}}$ which depends on $\mathcal{P}$ and
- a set of predicates pred which depends on type interpretations and consists of:
  - Variable predicate: each variable belongs to each type interpretation.
  - Saturation predicate (1): the contractum of a β-redex is in a type interpretation $\Rightarrow$ the β-redex is in the type interpretation.
  - Closure predicate (1): a term applied to a variable is in a type interpretation $\Rightarrow$ the term is in the set of terms given as parameter.

Ghilezan and Likavec claim that $\mathrm{pred}(\mathcal{P}) \Rightarrow \mathsf{SN} \subseteq \mathcal{P}$.

(where $\mathsf{SN} = \{M \mid \text{each reduction from } M \text{ is finite}\}$ = set of λ-terms typable in D).

Ghilezan and Likavec's Method [GL02]

full method - basic step

Recall that $\mathcal{P}$ is a set of terms. Using:

- a set of types $\tau \in \mathsf{Type}^2 ::= \alpha \mid \tau_1 \to \tau_2 \mid \tau_1 \cap \tau_2 \mid \Omega$,
- a type interpretation depending on $\mathcal{P}$,
- a set of predicates pred which depends on type interpretations and consists of:
  - Variable predicate: same as before.
  - Saturation predicate (2): similar to before.
  - Closure predicate (2): a term is in a type interpretation $\Rightarrow$ the abstraction of the term is in $\mathcal{P}$.
- an intersection type system (with omega and subtyping rule),

Ghilezan and Likavec prove that $\mathrm{pred}(\mathcal{P}) \Rightarrow T \subseteq \mathcal{P}$
where T is a set of typable terms under some restriction on types.

Ghilezan and Likavec's method [GL02]

full method- basic step continued

- It is not easy to prove $\mathrm{pred}(\mathcal{P})$. Hence, [GL02] introduces:
  ➤ stronger induction hypotheses. These are new predicates collected in a set $\mathrm{newpred}$.
  ➤ These new predicates do not deal with type interpretation
- $\mathrm{newpred}(\mathsf{CR})$ where
  $\mathsf{CR} = \{M \mid M \to_\beta^* M_1 \wedge M \to_\beta^* M_2 \Rightarrow \exists M'.\ M_1 \to_\beta^* M' \wedge M_2 \to_\beta^* M'\}$
- $\mathrm{newpred}(\mathsf{W})$ where
  $\mathsf{W} = \{M \mid \exists n \in \mathbb{N}.\ \exists x \in \mathcal{V}.\ \exists M, M_1, \ldots, M_n \in \Lambda.\ (M \to_\beta^* \lambda x.M \vee M \to_\beta^* xM_1 \ldots M_n)\}$ and
- $\mathrm{newpred}(\mathsf{S})$ where
  $\mathsf{S} = \{M \mid M \to_\beta^* M' \Rightarrow \exists N.\ M \to_h^* N \wedge N \to_i^* M'\}$ ($\to_h^*$ for head-reduction and $\to_i^*$ for internal-reduction

Ghilezan and Likavec's method [GL02]

full method- final step

- The final step of the method is to prove $\mathrm{newpred}(\mathcal{P}) \wedge \mathrm{Inv}(\mathcal{P}) \Rightarrow \Lambda = \mathcal{P}$ where Λ is the set of all the λ-terms and Invariance predicate Inv:
  If $M \in \Lambda$ then $\lambda x.M \in \mathcal{P} \iff M \in \mathcal{P}$.
- The authors give a set T of λ-terms that are typable in their type system with a type satisfying the necessary restrictions.
- This final step is done in two parts:
  - Let $M \in \Lambda$. Then:
    - $\lambda x.M \in T$
    - $\mathrm{newpred}(\mathcal{P}) \Rightarrow \lambda x.M \in \mathcal{P}$
  - $\mathrm{newpred}(\mathcal{P}) \wedge \mathrm{Inv}(\mathcal{P}) \Rightarrow M \in \mathcal{P}$
- $\mathrm{Inv}(\mathsf{CR})$ and $\mathrm{Inv}(\mathsf{S})$.

Ghilezan and Likavec's method fails
Counterexample

- Our paper lists in detail the problems with a number of lemas and proofs in [GL02].
- Here, we show one counterexample:

Claim [GL02]

$\mathrm{INV}(\mathcal{P}) \wedge \mathrm{VAR}(\mathcal{P}) \wedge \mathrm{SAT}(\mathcal{P}) \Rightarrow \Lambda = \mathcal{P}$.

Counter-example: $\mathrm{INV}(\mathsf{WN})$, $\mathrm{VAR}(\mathsf{WN})$ and $\mathrm{SAT}(\mathsf{WN})$ are true, but $\mathsf{WN} \neq \Lambda$.

Ghilezan and Likavec's method [GL02]

summary

First step:

- $\mathrm{pred1}(\mathcal{P}) \Rightarrow T \subseteq \mathcal{P}$.

(where T is a set of typable terms in a given type system)

Full method (false):

- $\mathrm{pred2}(\mathcal{P}) \Rightarrow \Lambda = \mathcal{P}$.

We tried to salvage the full method of Ghilezan and Likavec, but we failed. We did not go further than the basic step with $T = \mathsf{SN}$, which is a result Ghilezan and Likavec already proved.

Some similar proof methods have already been, as far as we know, successfully developed (for example by Gallier [Gal03]). However, they do not go further than the basic step and do not deal with Church-Rosser. Such methods can help in characterising typable terms w.r.t. a type system.

Koletsos and Stavrinos's method [KS08]

the outlines of their method

$$
\begin{tikzcd}
 & & \langle M, \mathcal{F} \rangle \arrow[ddll, "1"'] \arrow[d, "\Psi"] \arrow[ddrr, "1"] & & \\
 & & P \arrow[dl, "\beta"'] \arrow[dr, "\beta"] & & \\
\langle M_1, \mathcal{F}_1 \rangle \arrow[r, "\Psi"] \arrow[ddrr, "1"'] & P_1 \arrow[dr, "\beta"'] & \mathrm{CR} & P_2 \arrow[dl, "\beta"] & \langle M_2, \mathcal{F}_2 \rangle \arrow[l, "\Psi"'] \arrow[ddll, "1"] \\
 & & P_3 & & \\
 & & \langle M_3, \mathcal{F}_3 \rangle \arrow[u, "\Psi"'] & &
\end{tikzcd}
$$

An Extension of Koletsos and Stavrinos's method [KS08]

the central part

- Koletsos and Stavrinos's method [KS08] proves Church Rosser of β-reduction.
- We extend Koletsos and Stavrinos's method to prove Church Rosser of $\beta\eta$-reduction.
- $\mathrm{CRBE} = \{M \mid M \rightarrow^*_{\beta\eta} M_1 \wedge M \rightarrow^*_{\beta\eta} M_2 \Rightarrow \exists M'.\ M_1 \rightarrow^*_{\beta\eta} M' \wedge M_2 \rightarrow^*_{\beta\eta} M'\}$
- Using:
  - a set of types,
  - a type system,
  - a type interpretation based on CRBE and
  - a language typable in the type system,

  we prove that each term in the defined language is in CRBE.

An Extension of Koletsos and Stavrinos's method [KS08]

a bit of technicality

What is this new language? the parametrised language $\Lambda\eta_c \subseteq \Lambda$ is defined as follows:

1. If x is a variable distinct from c then
   - $x \in \Lambda\eta_c$.
   - If $M \in \Lambda\eta_c$ then $\lambda x.(M[x := c(cx)]) \in \Lambda\eta_c$.
   - If $Nx \in \Lambda\eta_c$, $x \notin \mathrm{fv}(N)$ and $N \neq c$ then $\lambda x.Nx \in \Lambda\eta_c$.
2. If $M, N \in \Lambda\eta_c$ then $cMN \in \Lambda\eta_c$.
3. If $M, N \in \Lambda\eta_c$ and M is a λ-abstraction then $MN \in \Lambda\eta_c$.
4. If $M \in \Lambda\eta_c$ then $cM \in \Lambda\eta_c$.

An Extension of Koletsos and Stavrinos's method [KS08]

a bit a technicality

$p \in \text{Path} ::= 0 \mid 1.p \mid 2.p.$

We define $M|_p$ as follows:

- $M|_0 = M$
- $(\lambda x.M)|_{1.p} = M|_p$
- $(MN)|_{1.p} = M|_p$
- $(MN)|_{2.p} = N|_p$.

Example: $(\lambda x.zx)|_{1.2.0} = (zx)|_{2.0} = x|_0 = x.$

An Extension of Koletsos and Stavrinos's method [KS08]

a bit a technicality

Let us define the three following common relations:

- $\beta ::= \langle (\lambda x.M)N, M[x := N] \rangle$
- $\eta ::= \langle \lambda x.Mx, M \rangle$, where $x \notin FV(M)$
- $\beta\eta = \beta \cup \eta$

Let $r \in \{\beta, \eta, \beta\eta\}$
$\mathcal{R}^r = \{L \mid \langle L, R \rangle \in r\}$ and $\mathcal{R}^r_M = \{p \mid M|_p \in \mathcal{R}^r\}$

Example: $\mathcal{R}^{\beta\eta}_{(\lambda x.yx)y} = \{0, 1.0\}$.

We define the ternary relation $\rightarrow_r$ as follows:

- $M \xrightarrow{0}_r M'$ if $\langle M, M' \rangle \in r$
- $\lambda x.M \xrightarrow{1.p}_r \lambda x.M'$ if $M \xrightarrow{p}_r M'$
- $MN \xrightarrow{1.p}_r M'N$ if $M \xrightarrow{p}_r M'$
- $NM \xrightarrow{2.p}_r NM'$ if $M \xrightarrow{p}_r M'$

$M \rightarrow_r M'$ if there exists p such that $M \xrightarrow{p}_r M'$.

Example: $(\lambda x.x)y \xrightarrow{0}_\beta y \Rightarrow \lambda y.(\lambda x.x)y \xrightarrow{1.0}_\beta \lambda y.y$.

An Extension of Koletsos and Stavrinos's method [KS08]

a bit a technicality - An erasure function

Erasure on terms:

- $|x|^c = x$
- $|\lambda x.N|^c = \lambda x.|N|^c$, if $x \neq c$
- $|cP|^c = |P|^c$
- $|NP|^c = |N|^c|P|^c$, if $N \neq c$

Example: $|(c(\lambda x.yx))y|^c = (\lambda x.yx)y$.

Erasure on paths:

- $|\langle M, 0\rangle|^c = 0$
- $|\langle \lambda x.M, 1.p\rangle|^c = 1.|\langle M, p\rangle|^c$, if $x \neq c$
- $|\langle MN, 1.p\rangle|^c = 1.|\langle M, p\rangle|^c$
- $|\langle cM, 2.p\rangle|^c = |\langle M, p\rangle|^c$
- $|\langle NM, 2.p\rangle|^c = 2.|\langle M, p\rangle|^c$, if $N \neq c$

Example: $|\langle (c(\lambda x.yx))y, 1.2.0\rangle|^c = 1.0$.

An Extension of Koletsos and Stavrinos's method [KS08]

a bit a technicality - a function from $\Lambda \times 2^{\mathsf{Path}}$ to $2^{\Lambda\eta_c}$

Let $c \notin \mathrm{fv}(M)$ and $\mathcal{F} \subseteq \mathcal{R}_M^{\beta\eta}$.

1. If $M \in \mathcal{V} \setminus \{c\}$ then $\mathcal{F} = \varnothing$ and
   $\Psi^c(M, \mathcal{F}) = \{c^n(M) \mid n > 0\}$
   $\Psi_0^c(M, \mathcal{F}) = \{M\}$
2. If $M = \lambda x.N$ and $x \neq c$ and $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}_N^{\beta\eta}$ then:
   $$\Psi^c(M, \mathcal{F}) = \begin{cases} \{c^n(\lambda x.P[x := c(cx)]) \mid n \geq 0 \wedge P \in \Psi^c(N, \mathcal{F}')\} & \text{if } 0 \notin \mathcal{F} \\ \{c^n(\lambda x.N') \mid n \geq 0 \wedge N' \in \Psi_0^c(N, \mathcal{F}')\} & \text{otherwise} \end{cases}$$
   $$\Psi_0^c(M, \mathcal{F}) = \begin{cases} \{\lambda x.N'[x := c(cx)] \mid N' \in \Psi^c(N, \mathcal{F}')\} & \text{if } 0 \notin \mathcal{F} \\ \{\lambda x.N' \mid N' \in \Psi_0^c(N, \mathcal{F}')\} & \text{otherwise} \end{cases}$$
3. If $M = NP$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}_N^{\beta\eta}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq \mathcal{R}_P^{\beta\eta}$ then:
   $$\Psi^c(M, \mathcal{F}) = \begin{cases} \{c^n(cN'P') \mid n \geq 0 \wedge N' \in \Psi^c(N, \mathcal{F}_1) \wedge P' \in \Psi^c(P, \mathcal{F}_2)\} & \text{if } 0 \notin \mathcal{F} \\ \{c^n(N'P') \mid n \geq 0 \wedge N' \in \Psi_0^c(N, \mathcal{F}_1) \wedge P' \in \Psi^c(P, \mathcal{F}_2)\} & \text{otherwise} \end{cases}$$
   $$\Psi_0^c(M, \mathcal{F}) = \begin{cases} \{cN'P' \mid N' \in \Psi^c(N, \mathcal{F}_1) \wedge P' \in \Psi_0^c(P, \mathcal{F}_2)\} & \text{if } 0 \notin \mathcal{F} \\ \{N'P' \mid N' \in \Psi_0^c(N, \mathcal{F}_1) \wedge P' \in \Psi_0^c(P, \mathcal{F}_2) & \text{otherwise} \end{cases}$$

An Extension of Koletsos and Stavrinos's method [KS08]

illustration of this technicality

Example:

$\Psi^c((\lambda x.(\lambda y.M)x)N, \{1, 1.0, 1.1.0\}) =$
$\{c^n((\lambda x.(\lambda y.P[y := c(cy)])x)Q) \mid n \geq 0 \wedge P \in \Psi^c(M, \varnothing) \wedge Q \in \Psi^c(N, \varnothing)\} \subseteq \Lambda\eta_c$,
where $x \notin \mathrm{fv}(\lambda y.M)$.

Let $p = 1.0$ then $(\lambda x.(\lambda y.M)x)N \xrightarrow{p}_{\beta\eta} (\lambda y.M)N$.

Let $n \geq 0$, $P \in \Psi^c(M, \varnothing)$, $Q \in \Psi^c(N, \varnothing)$ and $p' = \overbrace{2.\ldots.2}^{n}.1.0$. Then:

- $P_0 = c^n((\lambda x.(\lambda y.P[y := c(cy)])x)Q) \xrightarrow{p'}_{\beta\eta} c^n((\lambda y.P[y := c(cy)])Q)$
- $|\langle P_0, p'\rangle|^c = |\langle P_0, 2^n.1.0\rangle|^c = p$
- $c^n((\lambda y.P[y := c(cy)])Q) \in \Psi^c((\lambda y.M)N, \{0\})$

Let $c \notin \mathrm{fv}(M)$ and $\mathcal{F} \subseteq \mathcal{R}_M^{\beta\eta}$.

- Let $p \in \mathcal{F}$ and $M \xrightarrow{p}_{\beta\eta} M'$. We call the unique $\mathcal{F}' \subseteq \mathcal{R}_{M'}^{\beta\eta}$, such that for all $N \in \Psi^c(M, \mathcal{F})$ there exist $N' \in \Psi^c(M', \mathcal{F}')$ and $p' \in \mathcal{R}_N^{\beta\eta}$ such that $N \xrightarrow{p'}_{\beta\eta} N'$ and $|\langle N, p' \rangle|^c = p$, the set of $\beta\eta$-**residuals of** $\mathcal{F}$ **in** M' **relative to** p.
- A one-step $\beta\eta$-development of $\langle M, \mathcal{F} \rangle$, denoted $\langle M, \mathcal{F} \rangle \rightarrow_{\beta\eta d} \langle M', \mathcal{F}' \rangle$, is a $\beta\eta$-reduction $M \xrightarrow{p}_{\beta\eta} M'$ where $p \in \mathcal{F}$ and $\mathcal{F}'$ is the set of $\beta\eta$-residuals of $\mathcal{F}$ in M' relative to p. A $\beta\eta$-**development** is the transitive closure of a one-step $\beta\eta$-development. We write $M \rightarrow_1 M'$ for the $\beta\eta$-development $\langle M, \mathcal{F} \rangle \rightarrow_{\beta\eta d}^* \langle M', \mathcal{F}' \rangle$.

Lemma

If $c \notin \mathrm{fv}(M)$, $M \rightarrow_1 M_1$ and $M \rightarrow_1 M_2$ then there exists M_3 such that $M_1 \rightarrow_1 M_3$ and $M_2 \rightarrow_1 M_3$.

The transitive reflexive closure of $\rightarrow_{\beta\eta}$ is equal to the transitive reflexive closure of $\rightarrow_1$. We are now able to prove the (non-strict) inclusion of Λ in CRBE and the equality between these sets:

Lemma

$c \notin \mathrm{fv}(M) \Rightarrow M \in \mathrm{CRBE}$.

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