Reducibility proofs in $\lambda$-calculi with intersection types

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By using reducibility, new, simple and general methods can be developed to prove properties of the $\lambda$-calculus.

In our paper:

- We review and find the flaws in one reducibility method of proofs of Church-Rosser, standardisation and weak head normalisation.
- We review, adapt and non trivially extend another reducibility method of proofs of Church-Rosser.
1. Ghilezan and Likavec’s method:
   - According to this method, a certain property of the \( \lambda \)-calculus is proved to hold, if that property satisfies a certain set of predicates.
   - Unfortunately, this method does not work. We give counterexamples.

2. Koletsos and Stavrinos’s method:
   - This method aims to prove the Church-Rosser property of the untyped \( \lambda \)-calculus by showing first that a typed \( \lambda \)-calculus is confluent and using this to show the confluence of developments.
   - We adapt this method to \( \beta I \)-reduction.
   - We extend (this is non trivial) this method to \( \beta \eta \)-reduction.
Ghilezan and Likavec’s Method [GL02]

Ghilezan and Likavec designed a general proof method schema.

The basic step of the method: if a set of λ-terms \( \mathcal{P} \) satisfies a defined set of predicates \( \text{pred} \) then it contains a certain set of typable λ-terms \( T \).

\[ \Rightarrow \text{pred}(\mathcal{P}) \Rightarrow T \subseteq \mathcal{P} \]

Extension of the basic step: if a set of λ-terms \( \mathcal{P} \) satisfies a defined set of predicates \( \text{pred} \) then it contains the whole set of λ-terms.

\[ \Rightarrow \text{pred}(\mathcal{P}) \Rightarrow \Lambda = \mathcal{P} \]
Below, $\mathcal{P}$ is a set of terms. Using:

- a set of types $\sigma \in \text{Type}^1 := \alpha \mid \sigma_1 \rightarrow \sigma_2 \mid \sigma_1 \cap \sigma_2$,
- a type interpretation function $\llbracket - \rrbracket^1_\mathcal{P}$ which depends on $\mathcal{P}$ and
- a set of predicates $\text{pred}$ which depends on type interpretations and consists of:
  - Variable predicate: each variable belongs to each type interpretation.
  - Saturation predicate (1): the contractum of a $\beta$-redex is in a type interpretation $\Rightarrow$ the $\beta$-redex is in the type interpretation.
  - Closure predicate (1): a term applied to a variable is in a type interpretation $\Rightarrow$ the term is in the set of terms given as parameter.

Ghilezan and Likavec claim that $\text{pred}(\mathcal{P}) \Rightarrow \text{SN} \subseteq \mathcal{P}$.
(Where $\text{SN} = \{ M \mid \text{each reduction from } M \text{ is finite} \} = \text{set of } \lambda\text{-terms typable in } D$).
Recall that \( \mathcal{P} \) is a set of terms. Using:

- a set of types \( \tau \in \text{Type}^2 ::= \alpha \mid \tau_1 \rightarrow \tau_2 \mid \tau_1 \cap \tau_2 \mid \Omega, \)
- a type interpretation depending on \( \mathcal{P}, \)
- a set of predicates \( \text{pred} \) which depends on type interpretations and consists of:
  - **Variable predicate**: same as before.
  - **Saturation predicate (2)**: similar to before.
  - **Closure predicate (2)**: a term is in a type interpretation \( \Rightarrow \) the abstraction of the term is in \( \mathcal{P}. \)
- an intersection type system (with omega and subtyping rule),

Ghilezan and Likavec prove that \( \text{pred}(\mathcal{P}) \Rightarrow T \subseteq \mathcal{P} \)
where \( T \) is a set of typable terms under some restriction on types.
It is not easy to prove \( \text{pred}(P) \). Hence, [GL02] introduces:

- stronger induction hypotheses. These are new predicates collected in a set \( \text{newpred} \).
- These new predicates do not deal with type interpretation

**newpred(CR)** where

\[
\text{CR} = \{ M \mid M \rightarrow^\beta M_1 \land M \rightarrow^\beta M_2 \Rightarrow \exists M'. M_1 \rightarrow^\beta M' \land M_2 \rightarrow^\beta M' \}
\]

**newpred(W)** where

\[
\text{W} = \{ M \mid \exists n \in \mathbb{N}. \exists x \in \mathcal{V}. \exists M, M_1, \ldots, M_n \in \Lambda. \ (M \rightarrow^\beta \lambda x. M \lor M \rightarrow^\beta xM_1 \ldots M_n) \} \quad \text{and}
\]

**newpred(S)** where

\[
\text{S} = \{ M \mid M \rightarrow^\beta M' \Rightarrow \exists N. M \rightarrow^*_h N \land N \rightarrow^*_i M' \} \quad (\rightarrow^*_h \text{ for head-reduction and } \rightarrow^*_i \text{ for internal-reduction})
\]
The final step of the method is to prove
\[ \text{newpred}(\mathcal{P}) \land \text{Inv}(\mathcal{P}) \Rightarrow \Lambda = \mathcal{P} \]
where \( \Lambda \) is the set of all the \( \lambda \)-terms and
Invariance predicate \( \text{Inv} \):
If \( M \in \Lambda \) then \( \lambda x. M \in \mathcal{P} \iff M \in \mathcal{P} \).

The authors give a set \( T \) of \( \lambda \)-terms that are typable in their type system with a type satisfying the necessary restrictions.

This final step is done in two parts:
1. Let \( M \in \Lambda \). Then:
   1. \( \lambda x. M \in T \)
   2. \( \text{newpred}(\mathcal{P}) \Rightarrow \lambda x. M \in \mathcal{P} \)
   3. \( \text{newpred}(\mathcal{P}) \land \text{Inv}(\mathcal{P}) \Rightarrow M \in \mathcal{P} \)
2. \( \text{Inv(CR)} \) and \( \text{Inv(S)} \).
Our paper lists in detail the problems with a number of lemmas and proofs in [GL02].

Here, we show one counterexample:

Claim [GL02]

\[ \text{INV}(\mathcal{P}) \land \text{VAR}(\mathcal{P}) \land \text{SAT}(\mathcal{P}) \Rightarrow \Lambda = \mathcal{P}. \]

Counter-example: INV(WN), VAR(WN) and SAT(WN) are true, but WN \( \neq \Lambda \).
Ghilezan and Likavec’s method [GL02]

summary

First step:

- \( \text{pred1}(\mathcal{P}) \Rightarrow T \subseteq \mathcal{P} \).

(where \( T \) is a set of typable terms in a given type system)

Full method (false):

- \( \text{pred2}(\mathcal{P}) \Rightarrow \Lambda = \mathcal{P} \).

We tried to salvage the full method of Ghilezan and Likavec, but we failed. We did not go further than the basic step with \( T = \text{SN} \), which is a result Ghilezan and Likavec already proved.

Some similar proof methods have already been, as far as we know, successfully developed (for example by Gallier [Gal03]). However, they do not go further than the basic step and do not deal with Church-Rosser. Such methods can help in characterising typable terms w.r.t. a type system.
Koletsos and Stavrinos’s method [KS08]
the outlines of their method
An Extension of Koletsos and Stavrinos’s method [KS08]  
the central part

- Koletsos and Stavrinos’s method [KS08] proves Church Rosser of $\beta$-reduction.
- We extend Koletsos and Stavrinos’s method to prove Church Rosser of $\beta\eta$-reduction.

$$\text{CRBE} = \{ M \mid M \rightarrow^{*}_{\beta\eta} M_1 \land M \rightarrow^{*}_{\beta\eta} M_2 \Rightarrow \exists M'. \ M_1 \rightarrow^{*}_{\beta\eta} M' \land M_2 \rightarrow^{*}_{\beta\eta} M' \}$$

- Using:
  - a set of types,
  - a type system,
  - a type interpretation based on CRBE and
  - a language typable in the type system,

we prove that each term in the defined language is in CRBE.
What is this new language? the parametrised language $\Lambda \eta_c \subseteq \Lambda$ is defined as follows:

1. If $x$ is a variable distinct from $c$ then
   - $x \in \Lambda \eta_c$.
   - If $M \in \Lambda \eta_c$ then $\lambda x. (M[x := c(cx)]) \in \Lambda \eta_c$.
   - If $N x \in \Lambda \eta_c$, $x \notin \text{fv}(N)$ and $N \neq c$ then $\lambda x. N x \in \Lambda \eta_c$.

2. If $M, N \in \Lambda \eta_c$ then $cMN \in \Lambda \eta_c$.

3. If $M, N \in \Lambda \eta_c$ and $M$ is a $\lambda$-abstraction then $MN \in \Lambda \eta_c$.

4. If $M \in \Lambda \eta_c$ then $cM \in \Lambda \eta_c$. 
An Extension of Koletsos and Stavrinos’s method [KS08]
a bit a technicality

\[ p \in \text{Path} ::= 0 \mid 1.p \mid 2.p. \]

We define \( M|_p \) as follows:

- \( M|_0 = M \)
- \( (\lambda x. M)|_{1.p} = M|_p \)
- \( (MN)|_{1.p} = M|_p \)
- \( (MN)|_{2.p} = N|_p \).

Example: \( (\lambda x.zx)|_{1.2.0} = (zx)|_{2.0} = x|_0 = x. \)
An Extension of Koletsos and Stavrinos’s method [KS08]
a bit a technicality

Let us define the three following common relations:

▶ $\beta ::= \langle (\lambda x. M) N, M[x := N] \rangle$

▶ $\eta ::= \langle \lambda x. M x, M \rangle$, where $x \notin FV(M)$

▶ $\beta \eta = \beta \cup \eta$

Let $r \in \{ \beta, \eta, \beta \eta \}$

$\mathcal{R}^r = \{ L | \langle L, R \rangle \in r \}$ and $\mathcal{R}^r_M = \{ p | M|_p \in \mathcal{R}^r \}$

Example: $\mathcal{R}^{\beta \eta}_{(\lambda x. y x)y} = \{ 0, 1.0 \}$.

We define the ternary relation $\rightarrow_r$ as follows:

▶ $M \xrightarrow{0} r M'$ if $\langle M, M' \rangle \in r$

▶ $\lambda x. M \xrightarrow{1, p} r \lambda x. M'$ if $M \xrightarrow{p} r M'$

▶ $MN \xrightarrow{1, p} r M'N$ if $M \xrightarrow{p} r M'$

▶ $NM \xrightarrow{2, p} r NM'$ if $M \xrightarrow{p} r M'$

▶ $M \xrightarrow{r} M'$ if there exists $p$ such that $M \xrightarrow{p} r M'$.

Example: $(\lambda x. x)y \xrightarrow{0} \beta y \Rightarrow \lambda y.(\lambda x. x)y \xrightarrow{1.0} \beta \lambda y.y.$
An Extension of Koletsos and Stavrinos’s method [KS08]
a bit a technicality - An erasure function

Erasure on terms:

- \(|x|^c = x\)
- \(|\lambda x.N|^c = \lambda x.|N|^c\), if \(x \neq c\)
- \(|cP|^c = |P|^c\)
- \(|NP|^c = |N|^c|P|^c\), if \(N \neq c\)

Example: \(|(c(\lambda x.xy))y|^c = (\lambda x.yx)y|\).

Erasure on paths:

- \(|\langle M, 0 \rangle|^c = 0\)
- \(|\langle \lambda x.M, 1.p \rangle|^c = 1.|\langle M, p \rangle|^c\), if \(x \neq c\)
- \(|\langle MN, 1.p \rangle|^c = 1.|\langle M, p \rangle|^c\)
- \(|\langle cM, 2.p \rangle|^c = |\langle M, p \rangle|^c\)
- \(|\langle NM, 2.p \rangle|^c = 2.|\langle M, p \rangle|^c\), if \(N \neq c\)

Example: \(|\langle (c(\lambda x.yx))y, 1.2.0 \rangle|^c = 1.0.|\).
An Extension of Koletsos and Stavrinos’s method [KS08]
a bit a technicality - a function from $\Lambda \times 2^{\text{Path}}$ to $2^{\Lambda^c}$

Let $c \notin \text{fv}(M)$ and $\mathcal{F} \subseteq \mathcal{R}_M^{\beta \eta}$.

1. If $M \in \mathcal{V} \setminus \{c\}$ then $\mathcal{F} = \emptyset$ and

   $\Psi^c(M, \mathcal{F}) = \{c^n(M) \mid n > 0\}$

   $\Psi^c_0(M, \mathcal{F}) = \{M\}$

2. If $M = \lambda x. N$ and $x \neq c$ and $\mathcal{F}' = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}_N^{\beta \eta}$ then:

   $\Psi^c(M, \mathcal{F}) = \begin{cases} \{c^n(\lambda x.P[x := c(cx)]) \mid n \geq 0 \land P \in \Psi^c(N, \mathcal{F}')\} & \text{if } 0 \notin \mathcal{F} \\ \{c^n(\lambda x.N') \mid n \geq 0 \land N' \in \Psi^c_0(N, \mathcal{F}')\} & \text{otherwise} \end{cases}$

   $\Psi^c_0(M, \mathcal{F}) = \begin{cases} \{\lambda x.N'[x := c(cx)] \mid N' \in \Psi^c(N, \mathcal{F}')\} & \text{if } 0 \notin \mathcal{F} \\ \{\lambda x.N' \mid N' \in \Psi^c_0(N, \mathcal{F}')\} & \text{otherwise} \end{cases}$

3. If $M = NP$, $\mathcal{F}_1 = \{p \mid 1.p \in \mathcal{F}\} \subseteq \mathcal{R}_N^{\beta \eta}$ and $\mathcal{F}_2 = \{p \mid 2.p \in \mathcal{F}\} \subseteq \mathcal{R}_P^{\beta \eta}$ then:

   $\Psi^c(M, \mathcal{F}) = \begin{cases} \{c^n(cN' P') \mid n \geq 0 \land N' \in \Psi^c(N, \mathcal{F}_1) \land P' \in \Psi^c(P, \mathcal{F}_2)\} & \text{if } 0 \notin \mathcal{F} \\ \{c^n(N' P') \mid n \geq 0 \land N' \in \Psi^c_0(N, \mathcal{F}_1) \land P' \in \Psi^c(P, \mathcal{F}_2)\} & \text{otherwise} \end{cases}$

   $\Psi^c_0(M, \mathcal{F}) = \begin{cases} \{cN' P' \mid N' \in \Psi^c(N, \mathcal{F}_1) \land P' \in \Psi^c_0(P, \mathcal{F}_2)\} & \text{if } 0 \notin \mathcal{F} \\ \{N' P' \mid N' \in \Psi^c_0(N, \mathcal{F}_1) \land P' \in \Psi^c_0(P, \mathcal{F}_2)\} & \text{otherwise} \end{cases}$
An Extension of Koletos and Stavrinos’s method [KS08]
illustration of this technicality

Example:

\[ \Psi^c((\lambda x.(\lambda y. M)x)N, \{1, 1.0, 1.1.0\}) = \{c^n((\lambda x.(\lambda y. P[y := c(\text{cy}])x)Q) \mid n \geq 0 \land P \in \Psi^c(M, \emptyset) \land Q \in \Psi^c(N, \emptyset) \} \subseteq \Lambda \eta_c, \]

where \( x \notin \text{fv}(\lambda y. M) \).

Let \( p = 1.0 \) then \( (\lambda x.(\lambda y. M)x)N \xrightarrow{p} (\lambda y. M)N \).

Let \( n \geq 0, P \in \Psi^c(M, \emptyset), Q \in \Psi^c(N, \emptyset) \) and \( p' = 2 \ldots 2.1.0 \). Then:

- \( P_0 = c^n((\lambda x.(\lambda y. P[y := c(\text{cy}])x)Q) \xrightarrow{p'}_{\beta \eta} c^n((\lambda y. P[y := c(\text{cy})])Q) \)
- \( |\langle P_0, p' \rangle|_c = |\langle P_0, 2^n.1.0 \rangle|_c = p \)
- \( c^n((\lambda y. P[y := c(\text{cy})])Q) \in \Psi^c((\lambda y. M)N, \{0\}) \)
An Extension of Koletsos and Stavrinos’s method [KS08]

\( \beta\eta \)-developments

Let \( c \notin \text{fv}(M) \) and \( \mathcal{F} \subseteq \mathcal{R}_M^{\beta\eta} \).

- Let \( p \in \mathcal{F} \) and \( M \xrightarrow{p}^{\beta\eta} M' \). We call the unique \( \mathcal{F}' \subseteq \mathcal{R}_{M'}^{\beta\eta} \), such that for all \( N \in \Psi^c(M, \mathcal{F}) \) there exist \( N' \in \Psi^c(M', \mathcal{F}') \) and \( p' \in \mathcal{R}_{N'}^{\beta\eta} \) such that \( N \xrightarrow{p'}^{\beta\eta} N' \) and \( |\langle N, p' \rangle|^c = p \), the set of \( \beta\eta \)-residuals of \( \mathcal{F} \) in \( M' \) relative to \( p \).

- A one-step \( \beta\eta \)-development of \( \langle M, \mathcal{F} \rangle \), denoted \( \langle M, \mathcal{F} \rangle \xrightarrow{\beta\eta} \langle M', \mathcal{F}' \rangle \), is a \( \beta\eta \)-reduction \( M \xrightarrow{p}^{\beta\eta} M' \) where \( p \in \mathcal{F} \) and \( \mathcal{F}' \) is the set of \( \beta\eta \)-residuals of \( \mathcal{F} \) in \( M' \) relative to \( p \). A \( \beta\eta \)-development is the transitive closure of a one-step \( \beta\eta \)-development. We write \( M \rightarrow_1 M' \) for the \( \beta\eta \)-development \( \langle M, \mathcal{F} \rangle \rightarrow^{*\eta} \langle M', \mathcal{F}' \rangle \).

Lemma

If \( c \notin \text{fv}(M) \), \( M \rightarrow_1 M_1 \) and \( M \rightarrow_1 M_2 \) then there exists \( M_3 \) such that \( M_1 \rightarrow_1 M_3 \) and \( M_2 \rightarrow_1 M_3 \).
The transitive reflexive closure of $\rightarrow_{\beta\eta}$ is equal to the transitive reflexive closure of $\rightarrow_1$. We are now able to prove the (non-strict) inclusion of $\Lambda$ in CRBE and the equality between these sets:

**Lemma**

$c \not\in \text{fv}(M) \Rightarrow M \in \text{CRBE}$. 
J. Gallier.
Typing untyped $\lambda$-terms, or reducibility strikes again!.

S. Ghilezan and S. Likavec.
Reducibility: A ubiquitous method in lambda calculus with intersection types.

G. Koletsos and G. Stavrinos.
Church-Rosser property and intersection types.
To appear.